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THESIS

DEFENSE OF THE SEA BASE - AN ANALYTICAL MODEL

by

Henry S. Kim

December 2003

Thesis Advisor:
Second Reader:

Alan R. Washburn
Moshe Kress

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The model also gives a commander insight into the optimal placement of defenders with respect to parameters such as threat sector, minimum detection range, attacker and defender velocity, and defender weapon ranges. The model can also be used for Operational Requirements (ORs) development by Sea Base system designers. By inputting parameters associated with certain scenarios, system developers can see how performance of a specific parameter, such as weapons range, probability of kill, and radar detection range, can affect the quality of Sea Base defense with respect to the effective area of defender coverage and the number of defenders required to achieve a certain level of protection.

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DEFENSE OF THE SEA BASE – AN ANALYTICAL MODEL

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B.A., University of California, Los Angeles, 1996

Submitted in partial fulfillment of the
requirements for the degree of

MASTER SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
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TABLE OF CONTENTS

I.	INTRODUCTION.....	1
A.	BACKGROUND AND PURPOSE OR RESEARCH.....	1
II.	PARAMETERS AND BASIC ASSUMPTIONS.....	3
A.	PARAMETERS.....	3
B.	ASSUMPTIONS.....	3
III.	MODEL FORMULATION: 1 (HVU) x 1(DEFENDER) x 1(ATTACKER).....	5
A.	PROBABILITY OF KILL.....	5
B.	CRITICAL DISTANCE (cd)	6
C.	PRE-ENGAGEMENT MANEUVERS	10
D.	THREAT SECTOR.....	11
IV.	MODEL FORMULATION: m (HVU) x 1(DEFENDER) x 1(ATTACKER)	17
A.	EFFECT OF MULTIPLE HVUS ON THREAT SECTOR	17
B.	HVU LATERAL SEPARATION (ls).....	19
V.	MODEL FORMULATION: 1 (HVU) x 1(DEFENDER) x p(ATTACKERS)	23
A.	MAXIMUM SALVO NUMBER CALCULATION	23
1.	Effects of Weapon Flight Time.....	23
2.	Battle Damage Assessment.....	27
B.	HDT PROBABILITY OF KILL WITH MARKOV CHAINS.....	32
C.	TYPE I AND TYPE II ERRORS	33
D.	MISSILE FIRING POLICIES AGAINST HDTs.....	34
1.	Broad vs. Narrow Beam FC Radar.....	36
2.	Narrow Beam Model.....	37
3.	Wide Beam Model.....	39
4.	Narrow vs. Wide Beam Model Comparison.....	41
a.	<i>Narrow Beam Model Example</i>	41
b.	<i>Wide Beam Model Example</i>	43
VI.	NUMERICAL EXAMPLES	45
A.	DETERMINING MAX THREAT SECTOR.....	45
B.	DETERMINING PROBABILITY OF KILL (SINGLE ATTACKER)....	46
1.	Effect Of Kill Probability on Threat Sector.....	47
C.	EFFECT OF DETECTION RANGE.....	49
D.	EFFECT OF LATERAL SEPARATION AMONG HVUS.....	50
E.	DETERMING SALVO REQUIREMENTS.....	51
VII.	MODEL LIMITATIONS AND AREAS FOR FURTHER ANALYSIS	53
A.	DIRECT OVERFLIGHT OF ATTACKERS	53
B.	CALCULATIONS FOR MULTIPLE DEFENDERS	53
C.	MOVING HVUS.....	55
VIII.	CONCLUSION	57
IX.	APPENDIX A - EXPECTED VALUE AND SIMULATION CODE	59

LIST OF REFERENCES.....	67
INITIAL DISTRIBUTION LIST	69

LIST OF FIGURES

Figure 1.	Engagement Range.....	5
Figure 2.	Increased Engagement Range.....	7
Figure 3.	Engagement Zone.....	8
Figure 4.	Critical Distance.....	9
Figure 5.	Defender's Travel Distance.....	11
Figure 6.	Defender Placement Zone.....	12
Figure 7.	Defender Placement Zone Calculations.....	13
Figure 8.	Maximum Sector Defender Placement.....	14
Figure 9.	Multiple HVUs.....	17
Figure 10.	Effective Threat Sector.....	18
Figure 11.	Effective Threat Sector Calculations I.....	19
Figure 12.	Effective Threat Sector Calculations II.....	20
Figure 13.	Shadow Region.....	21
Figure 14.	Salvo Calculations.....	24
Figure 15.	Salvo Calculations with BDA.....	28
Figure 16.	Transition Matrix for Salvo Calculations.....	33
Figure 17.	Mk 92 Fire Control System.....	36
Figure 18.	Narrow Beam Engagement.....	37
Figure 19.	Wide Beam Engagement.....	39
Figure 20.	Recommended Salvo Size (x) (Narrow Beam).....	41
Figure 21.	Recommended Salvo Size (x) (Narrow Beam).....	42
Figure 22.	Recommended Salvo Size (x) (Narrow Beam).....	42
Figure 23.	Recommended Salvo Size (x) (Narrow Beam).....	42
Figure 24.	Recommended Salvo Size (x) (Wide Beam).....	43
Figure 25.	Recommended Salvo Size (x) (Wide Beam).....	43
Figure 26.	Recommended Salvo Size (x) (Wide Beam).....	44
Figure 27.	Recommended Salvo Size (x) (Wide Beam).....	44
Figure 28.	Separate Coverage Zones.....	54
Figure 29.	Combined Coverage Zone.....	55

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EXECUTIVE SUMMARY

Recently the Navy has been facing an increasing problem countering High Density Threats. Current tactics and weapon systems are focused on relatively small raids. In addition, many models are centered around a single High Value Unit (HVU) in a Carrier Battle Group (CVBG) and may be inadequate when considering the defense of a Sea Base with multiple HVUs. Moreover, with the Navy reaching its smallest fleet size ever since World War II, efficient use of limited defending assets such as guided missile destroyers and cruisers has become critical if the Department of the Navy is to continue to shape the political-military environment on a global scale. The goal in this thesis is to examine the critical factors that contribute to the defense of the Sea Base with respect to multiple HVUs against HDTs. The paper will focus primarily on the following factors:

- (1) defender placement relative to threat and HVU positions,
- (2) defender weapons speed,
- (3) defender weapons inventory,
- (4) defender weapons range,
- (5) defender weapons single shot probability of kill,
- (6) HDT size,
- (7) HDT speed,
- (8) threat sector size,
- (9) number of HVUs, and
- (10) placement of HVUs relative to threat and defenders positions.

The analytical model provides an understanding of the relationships of these factors and allows Systems Engineers to perform trade off studies during ship or weapon system development. In addition, the model can also be used by an operational commander charged with the Sea Base defense to utilize his given assets in the most efficient manner.

The study is divided into three phases, each building on the previous phase. Phase I considers the scenario with a single HVU, a single defender, and a single attacker. This phase describes the fundamental relationships that comprise Sea Base defense and concludes with the maximum sector a single defender can cover within the constraints of

this scenario. Phase II explores the effect of having multiple HVUs on defense of the Sea Base with particular emphasis on threat sector coverage and HVU placement. Finally, Phase III looks at the effect of having multiple attackers (HDT). This phase examines current capabilities and limitations of weapon systems and tactics against HDTs through a comparative analysis with a modified weapons system and tactic. The modified system is a Fire Control (FC) radar designed for a wide beam illumination of multiple targets. In addition, the modified system envisions increasing current missile salvo sizes beyond two, as prescribed by the shoot-shoot-look-shoot policy, to allow greater expected number of kills against the HDT.

The study concludes with numerical examples that illustrate the ideas developed in this paper and suggested areas for further study.

I. INTRODUCTION

A. BACKGROUND AND PURPOSE OF RESEARCH

Arquilla and Ronfeldt describe the history of military warfare as a progressive development of four basic types of engagement: melee, massing, maneuver, and swarming. Battles have advanced from disordered clashes of individuals or small groups, to the strategy of massed but highly structured formations, and then to the adoption of maneuver. The swarm tactic is argued to become the next major advance in military doctrine.¹ In addition, swarm tactics may be used unintentionally by an enemy. Iraq's extended-range Scud ballistic missiles were so poorly constructed that they broke up under the stress of reentry, effectively creating a swarm of "decoys" around the warhead that confused the guidance system of the Patriot antimissile defense system. As a result, few if any Patriot interceptions were successful.²

In response to the potential threat of swarm tactics, the Department of Defense has been conducting research to develop systems to counter such High Density Threats (HDTs). The Millennium ship self-defense system on Sea SLICE was used in Fleet Battle Experiment (FBE) J to prove out some key aspects of future littoral combat ship requirements, namely the "...[d]efeat of 'swarm attacks' of high speed armed intruder craft in both symmetric and asymmetric warfare environments."³

The purpose of this analysis is to develop an analytical model that describes defense for the Sea Base. Although models have been developed for defense of a carrier battle group (CVBG) with one High Value Unit (HVU) against air, surface and subsurface attacks, there are unique aspects of the Sea Base that are not specifically addressed in CVBG defense models. First, the defense of the sea base is different in that there are multiple HVUs (Expeditionary Warships – EXWAR Ships) expected in the Sea Base. In addition, there is a credible threat of being overwhelmed by HDTs in close proximity to enemy shores where the enemy need not have a large navy or long range air force to launch an assault. This model will specifically address the issue of defending

¹ John Arquilla and David Ronfeldt, *Swarming and the Future of Conflict* (Santa Monica, CA: RAND, 2000). Ch. 2

² Theodore A. Postol, "Lessons of the Gulf War Experience with Patriot," *International Security* 16 (3) (Winter 1991/1992), pp. 119-171.

³ <http://www.oerlikoncontraves.com/infoservice/aktuell/aktuell.htm>, November 23, 2003.

multiple HVUs against HDTs. The model will explore how HVU and defender positioning affect the quality of force protection with respect to various attackers, weapon, and sensor systems.

The analytical model is developed in phases, starting with one HVU, one defender, and one attacker unit and eventually progressing towards multiple HVUs, defenders, and attacker units. The model explores optimal placement of defenders with respect to parameters such as threat sector, minimum detection range, attacker and defender velocity, and defender weapon ranges.

The model can also be used for Operational Requirements (ORs) development by Sea Base system designers. By inputting parameters associated with certain scenarios, system developers can see how performance of a specific parameter, such as weapons range, probability of kill, and radar detection range, can affect the quality of Sea Base defense with respect to the effective area of defender coverage and the number of defenders required to achieve a certain level of protection. Additionally, if given a probability of success requirement for a defender against a certain number of attackers, the model can determine the number of shots/salvos which yields that level of confidence and in turn, give insight into what parameters play critical roles in achieving that level of success.

II. PARAMETERS AND BASIC ASSUMPTIONS

Parameters

- Attacker velocity (kts)
- Defender velocity (kts)
- Defender fire rate (rounds/hrs, λ)
- Defender weapons range (nm)
- Detection range (nm)
- Single-shot probability of kill given an engagement (p)
- Attacker Sector (2θ)

Assumptions

- HVUs stationary
- Attacker heads straight for HVU
- Attacker speed \gg Defender speed (i.e., missile/UCAV vs. ship)
- Changes in velocity are instantaneous
- Identification of enemy is instantaneous
- Perfect Battle Damage Assessment

Some of the aforementioned parameters are given values during model formulation below. As is shown, many of these parameters are interdependent. Thus, keeping some parameters constant allows the model to solve for the unknown parameters.

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III. MODEL FORMULATION: 1 (HVU) x 1(DEFENDER) x 1(ATTACKER)

A. PROBABILITY OF KILL

While providing protection for HVUs, a defender may want to stay as close to the HVUs as possible to allow flexibility in degree of sector coverage. A defender stationed right along side a HVU, for example, would theoretically be able to provide 360° coverage.

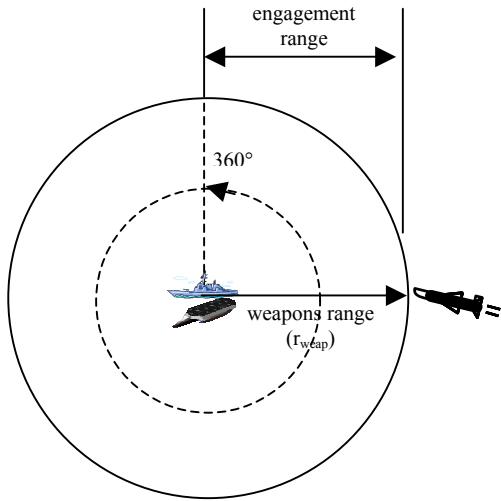


Figure 1 – Engagement Range

The defender then, presumably, has the length of the weapons range (r_{weap}) to engage incoming attackers. The amount of time an attacker is within the defender's weapons range is given by Eq [1]. The number of shots the defender can fire will be a function of the attacker's speed, the defender's weapons range, and the defender's rate of fire and is given by Eq [2]. If p is the single shot probability of a defender's weapon killing an attacker and $1-p$ is the probability of the attacker surviving, then the probability of an attacker surviving multiple independent shots in this scenario is given by Eq [3].

Variables

t = time attacker is within defender's weapons range

r_{weap} = defender's weapons range

v_{tar} = attacker velocity

n = number of shots fired by defender

λ = defender's firing rate

p = single shot probability of defender killing an attacker

Equations

$$t = r_{weap} / v_{tar} \quad \text{Eq [1]}$$

$$n = \lambda \times t \quad \text{Eq [2]}$$

$$P_k = [1 - (1 - p)^n] \quad \text{Eq [3]}$$

Probability of kill can also be considered stochastically if the defender's fire rate λ , is modeled as a Poisson process. In such a case, the probability of kill can be expressed as:

$$P_k = 1 - e^{-\lambda p t} \quad \text{Eq [4]}$$

For example, let:

$$\lambda = 1 \text{ round per 8 seconds}$$

$$p = 0.7$$

$$t = 30 \text{ seconds}$$

$$P_k = 1 - e^{-(1/8)(0.7)(30)}$$

$$= 0.9276$$

B. CRITICAL DISTANCE (cd)

In scenarios with respect to current US Navy capabilities, attackers are few in number (1 or 2 missiles or aircraft), defender's weapons range is several tens of nautical miles, and defender's single shot probability of kill is relatively high. Still the number of possible shots may not suffice to successfully kill all the attackers with a high level of probability, especially for supersonic attackers. This problem is exacerbated further by HDT attacks. Thus, the defender may want to somehow increase n , thereby improving its probability of successfully engaging the attacker(s). Although Eq [1] and [2] show that

increasing the rate of fire or weapons range increases n and could improve our probability of kill (Eq [4]), it is possible for a defender to lengthen its engagement range, without modification to its weapons systems, by positioning itself closer towards the attacker at the start of the engagement. The distance a defender can move towards an incoming attacker will be discussed in the section *Pre-engagement Maneuvers*. In the ideal case, the figure below shows that the engagement can be maximized to $2 \times r_{weap}$.

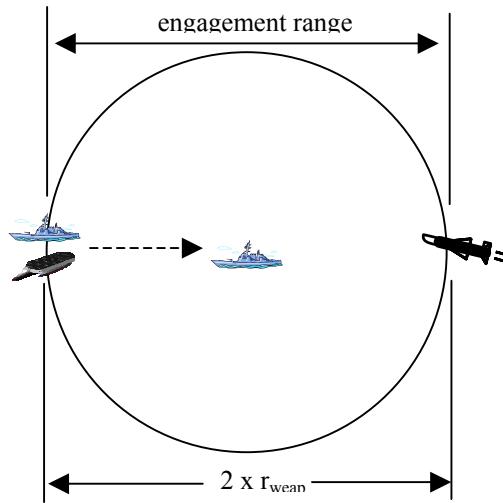


Figure 2 - Increased Engagement Range

We can modify Eq [1] to accommodate an increased engagement range as shown by Eq [5].

$$t = 2 \times r_{weap} / v_{tar} \quad \text{Eq [5]}$$

However, the defender need not stay stationary during the engagement. Indeed, we assume that the defender attempts to further maximize engagement time by paralleling the attacker course.

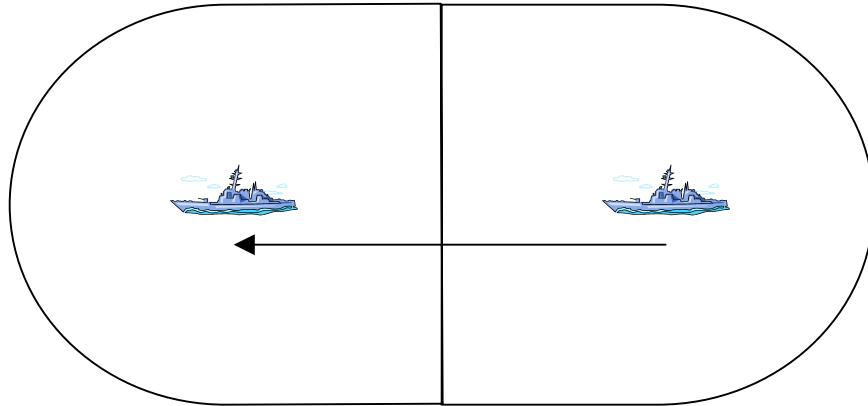


Figure 3 - Engagement Zone

Then the attacker crosses the engagement zone with a relative velocity given by Eq [6]. Substituting Eq [6] into Eq [5] yields Eq [7].

Variables

v_{def} = defender velocity

v_{rel} = relative velocity

Equation

$$v_{rel} = v_{tar} - v_{def} \quad \text{Eq [6]}$$

$$t_{max} = 2 \times r_{weap} / v_{rel} \quad \text{Eq [7]}$$

We will assume that $v_{def} < v_{tar}$, so $v_{rel} < v_{tar}$, since we assume that the defender is not stationary. The time variable in Eq [7] is subscripted with *max* since it is always larger than the time variable in Eq [5]. For ease of reference, we will also rewrite Eq [2] with t_{max} and label it Eq [8].

$$n = \lambda \times t_{max} \quad \text{Eq [8]}$$

Of course, if the attacker velocity is much greater than the defender velocity, this movement becomes insignificant. However, if the engagement range is sufficiently large, the movement of the defender along the attacker course becomes non-trivial even for

scenarios with surface ships as defenders and missiles as attackers. This movement will be referred to as *adjustment distance* (*ad*). As the following figure shows, the defender must then position itself no closer to the HVU than the *critical distance* (*cd*), in order to maximize its potential engagement range. The positions for the defender and attacker at this juncture shall be labeled *critical points of engagement*: (*cpe_{def}*) and (*cpe_{tar}*).

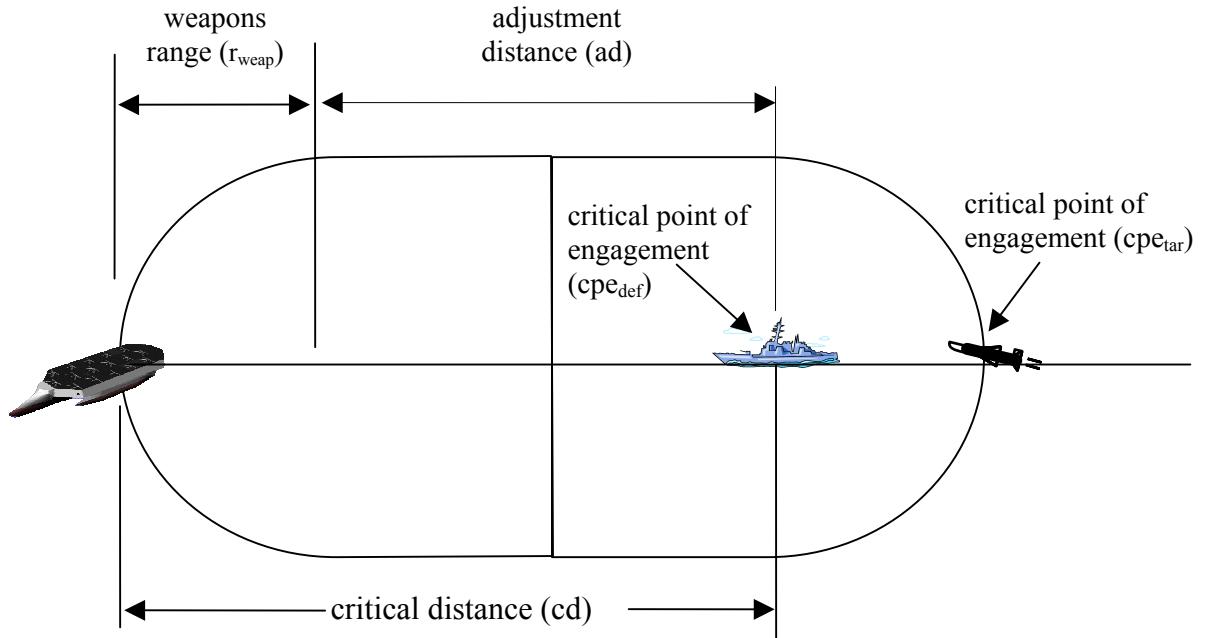


Figure 4 - Critical Distance

This critical distance is the sum of the defender's weapons range plus the defender's adjustment distance during the engagement.

$$cd = r_{weap} + ad \quad \text{Eq [9]}$$

The adjustment distance is a function of the defender's speed and the amount of time an attacker is within the defender's weapons range.

$$ad = v_{def} \times t_{max} \quad \text{Eq [10]}$$

Thus, substituting Eq [6], [7] and [10] into Eq [9] yields the following equation for critical distance:

$$\begin{aligned}
 cd &= r_{weap} + v_{def} [2r_{weap}/(v_{tar} - v_{def})] \\
 &= r_{weap} + 2r_{weap}/[(v_{tar}/v_{def}) - 1] \\
 &= r_{weap} [1 + 2 / (\mu - 1)] \quad \text{Eq [11]}
 \end{aligned}$$

Where μ is the velocity ratio v_{tar}/v_{def} .

C. PRE-ENGAGEMENT MANEUVERS

The next step is to consider how radar detection range can allow a defender time to maneuver into this critical point of engagement (cpe_{def}) prior to engagement. Consider the situation where radar coverage is centered around the HVU. If the attacker is detected at a certain detection range (r_{det}) from the HVU and the defender desires to be at the *critical point of engagement* (cpe_{def}) to maximize the engagement range, then the defender has the same amount of time to travel to cpe_{def} as the attacker has to travel to cpe_{tar} . This time (t_0) is a function of the attacker's relative velocity and the *attacker travel distance* (atd) from initial detection to cpe_{tar} and is given by Eq [13]. The atd can be determined by subtracting the engagement range (r_{engage}) from the detection range (r_{det}) as given by Eq [12].

$$\begin{aligned}
 r_{engage} &= cd + r_{weap} \\
 atd &= r_{det} - r_{engage} \quad \text{Eq [12]}
 \end{aligned}$$

$$t_0 = atd / v_{tar} \quad \text{Eq [13]}$$

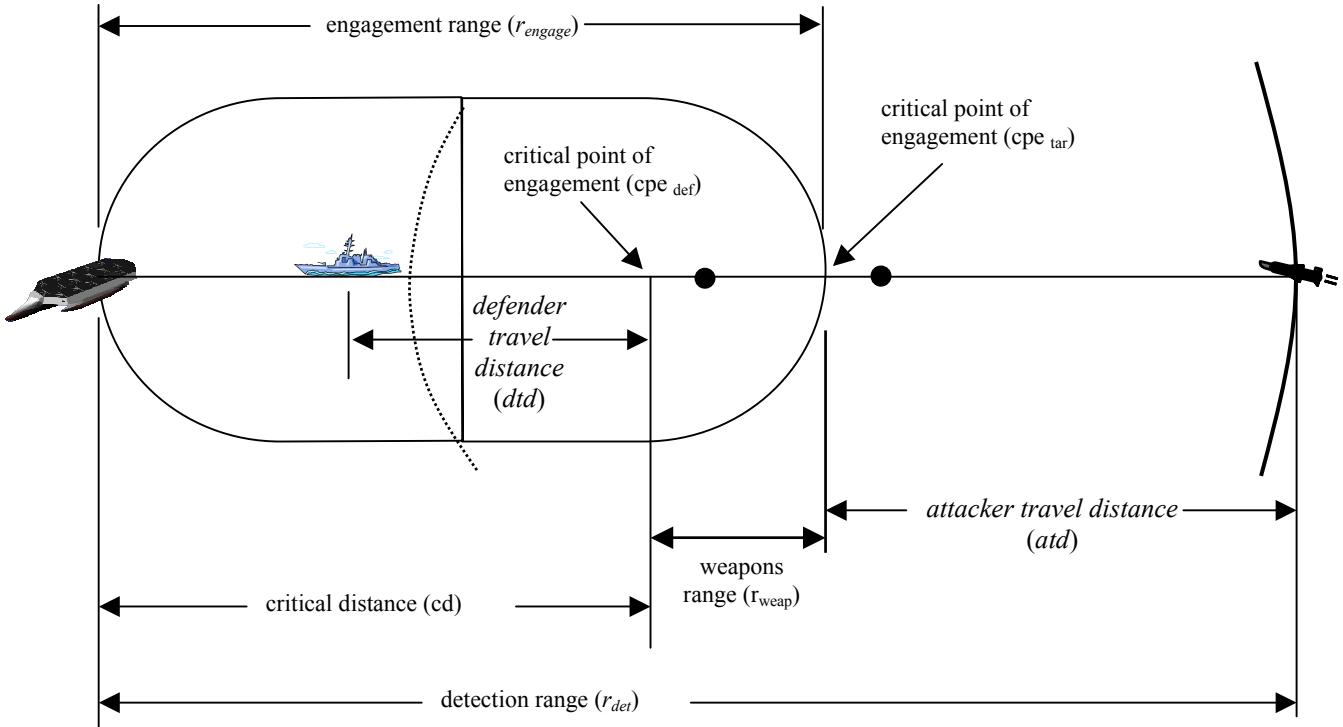


Figure 5 - Defender's Travel Distance

The *defender travel distance* (dtd) in this situation can be determined by multiplying t_0 and the defender's velocity v_{def} which yields the following equation:

$$dtd = t_0 \times v_{def} \quad \text{Eq [14]}$$

Hence, a commander must place a defender within a circle centered around cpe_{def} with radius dtd in order to be able to engage an inbound attacker at cpe_{tar} .

D. THREAT SECTOR

In the worst case where an attacker can approach the HVU from the outer edges of the threat sector, a defender must be in a position to reach the critical engagement points (pts 1 and 2) along the attacker's approach vector as depicted in the following figure. Since the defender can only be as far away from the critical engagement points 1 and 2 as determined by dtd , the defender should be placed in the intersection of the two circles with a radius dtd centered around points 1 and 2.

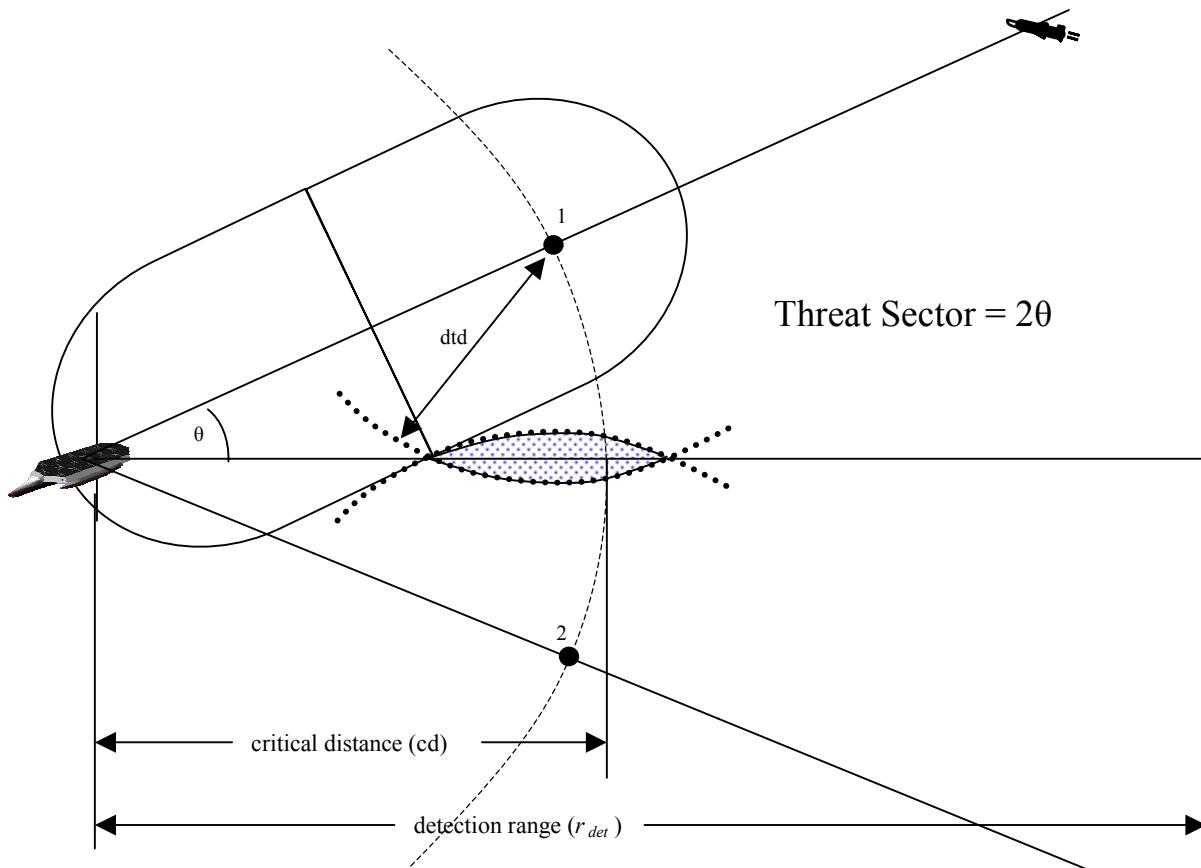


Figure 6 - Defender Placement Zone

In order to give a commander a frame of reference from which to place the defender in this overlap zone, we add the variables x , y , z and α to determine points a , b , c , and d by using some basic trigonometry. The shaded lens in Figure 7 is the approximate overlap area for defender placement.

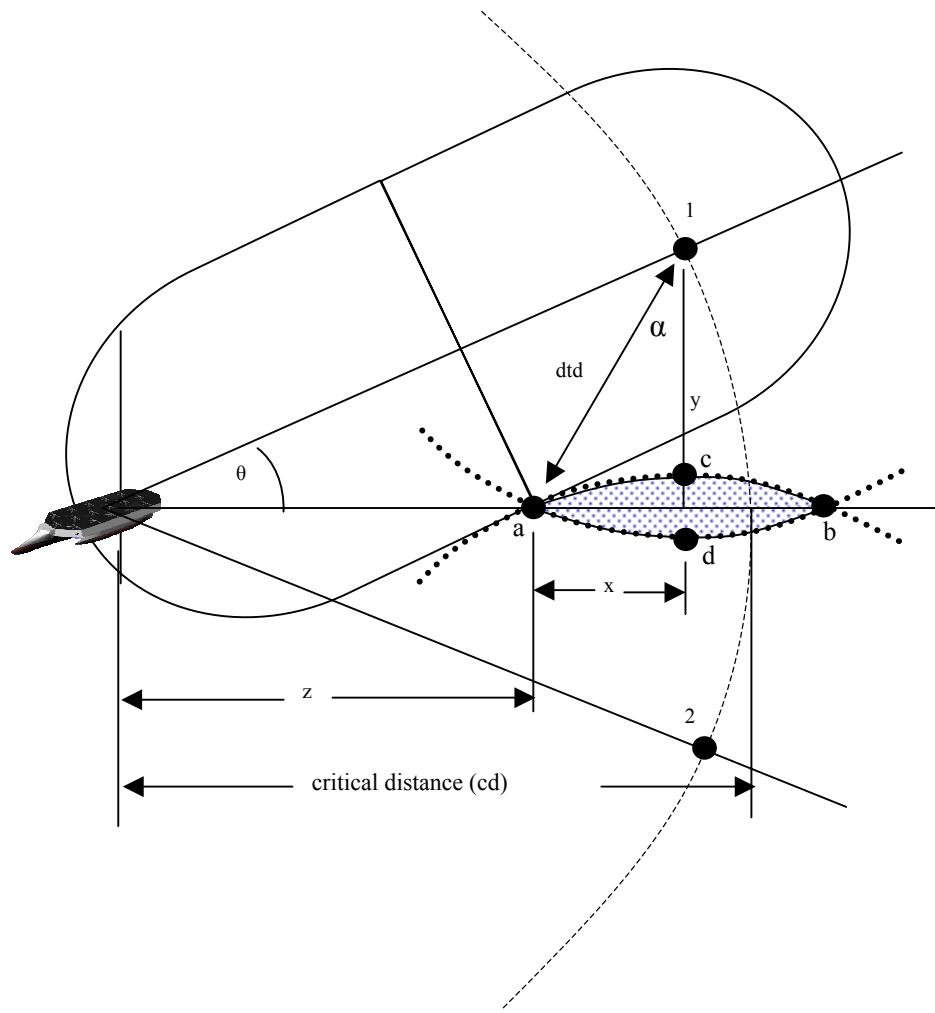


Figure 7 - Defender Placement Zone Calculations

$$y = cd \times \sin \theta \quad \text{Eq [15]}$$

$$\alpha = \cos^{-1} (y / dtd) \quad \text{Eq [16]}$$

$$x = \sin \alpha \times dtd \quad \text{Eq [17]}$$

$$z = (\cos \theta \times cd) - x \quad \text{Eq [18]}$$

Substituting Eq [16] and [17] into Eq [18] yields:

$$z = \cos \theta \times cd - \sin [\cos^{-1} (y / dtd)] \times dtd \quad \text{Eq [19]}$$

Thus, point *a* is distance *z* from the HVU along the center of the threat sector. Point *b* is $z + 2x$. Points *c* and *d* are perpendicular to the line segment formed by the points *a* and *b* and are both a distance $dtd - y$ away from the midpoint.

Holding other parameters constant, we can see that widening the threat sector will reduce the area of overlap. The maximum threat sector that can be accommodated is when the area of overlap converges to a point. This point shall be referred to with the Greek letter κ . Continuing to widen the threat sector further prevents the defender from being able to reach both cpe_{def} 1 or 2 from any initial placement point. Using the following figure for visual reference, we can see that the maximum threat sector is a function of the critical distance (cd) and the defender travel distance (dtd) and is given by Eq [20]. Point κ also lies on the line that divides the threat sector in half. The distance of κ from the HVU is a function of cd and dtd and is given by Eq [21].

$$\theta_{\max} = \sin^{-1} (dtd / cd) \quad \text{Eq [20]$$

$$\kappa = (cd^2 - dtd^2)^{1/2} \quad \text{Eq [21]}$$

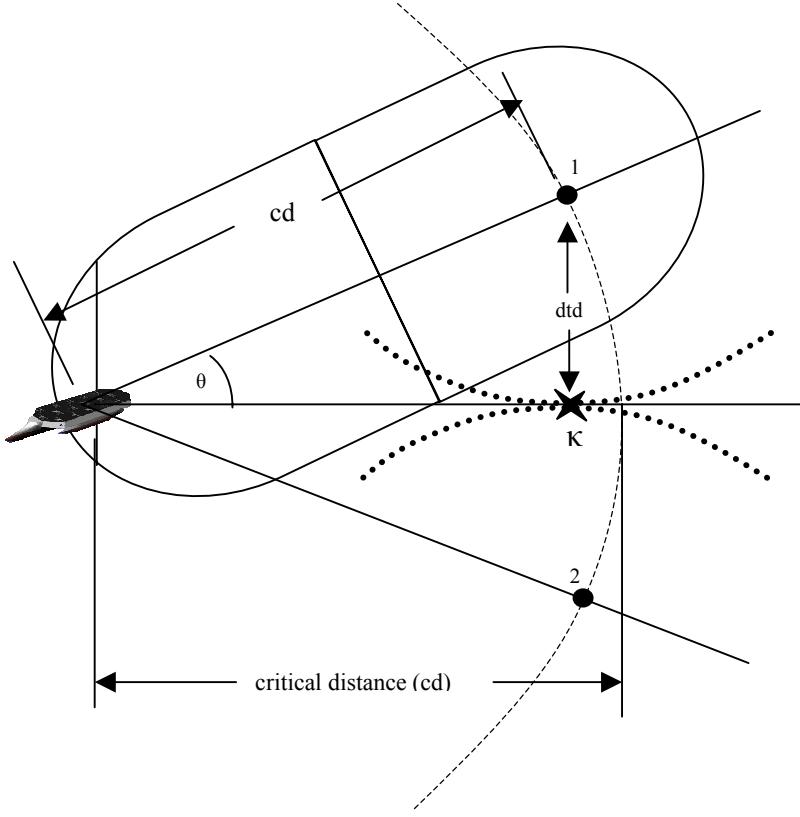


Figure 8 - Maximum Sector Defender Placement

If dtd equals or exceeds cd , then a single defender stationed alongside the HVU would be able to reach not only the critical distances at points 1 and 2, but would be able to provide 360° coverage.

Thus, given a threat sector, this model can generate an area for defender placement such that the defender can provide maximum effective coverage over the entire threat sector. In addition, given all other parameters except θ , the model can generate a maximum effective threat sector coverage for a single defender and the corresponding position κ for defender placement.

Example:

Given:

Attacker speed	750 kts
Defender speed	30 kts
Detection range	150 nm
Weapons range	25 nm

Using Eq [6]:

$$\begin{aligned} v_{rel} &= v_{tar} - v_{def} \\ &= 750 \text{ kts} - 30 \text{ kts} \\ &= 720 \text{ kts} \end{aligned}$$

Using Eq [7]:

$$\begin{aligned} t_{max} &= 2 \times r_{weap} / v_{rel} \\ &= 2 \times 25 \text{ nm} / 720 \text{ kts} \\ &= 0.0667 \text{ hrs} \end{aligned}$$

Using Eq [11]:

$$\begin{aligned} cd &= r_{weap} [1 + 2 / (\mu - 1)] \quad (\mu = v_{tar} / v_{def}) \\ &= 25 \text{ nm} [1 + 2 / ((750 \text{ kts} / 30 \text{ kts}) - 1)] \\ &= 27.01 \text{ nm} \end{aligned}$$

Using Eq [12]:

$$\begin{aligned} atd &= r_{det} - (cd + r_{weap}) \\ &= 150 \text{ nm} - (27.01 \text{ nm} + 25 \text{ nm}) \\ &= 97.99 \text{ nm} \end{aligned}$$

Using Eq [13]:

$$\begin{aligned}
 t_0 &= atd / v_{tar} \\
 &= 97.99\text{nm} / 750\text{kts} \\
 &= 0.1307 \text{ hrs}
 \end{aligned}$$

Using Eq [14]:

$$\begin{aligned}
 dtd &= t_0 \times v_{def} \\
 &= 0.1307 \text{ hrs} \times 30\text{kts} \\
 &= 3.92\text{nm}
 \end{aligned}$$

Using Eq [20]:

$$\begin{aligned}
 \theta_{\max} &= \sin^{-1} (dtd / cd) \\
 &= \sin^{-1} (3.92\text{nm} / 27.01\text{nm}) \\
 &= 8.344^\circ
 \end{aligned}$$

In this case, solving for θ_{\max} yields 8.344° or a threat sector coverage of approximately 16.5° .

IV. MODEL FORMULATION: m (HVU) x 1(DEFENDER) x 1(ATTACKER)

In this section, we explore the effects of multiple HVUs on a single defender's placement against a single attacker.

A. EFFECT OF MULTIPLE HVUS ON THREAT SECTOR

If a defender is already defending against its maximum threat sector, multiple HVUs must be placed in close proximity to one another for the defender to provide maximum protection. However, in the case for the Sea Base, EXWAR ships may be restricted in how close they may be stationed from one another because of the large airspace requirements inherent in heavy air operations. Thus, from the perspective of the approaching attacker, it is more than likely the attacker will find some angular separation (Ω) between HVUs. However, any angular offset of secondary HVUs from the primary HVU will effectively increase the maximum threat sector a defender must defend.

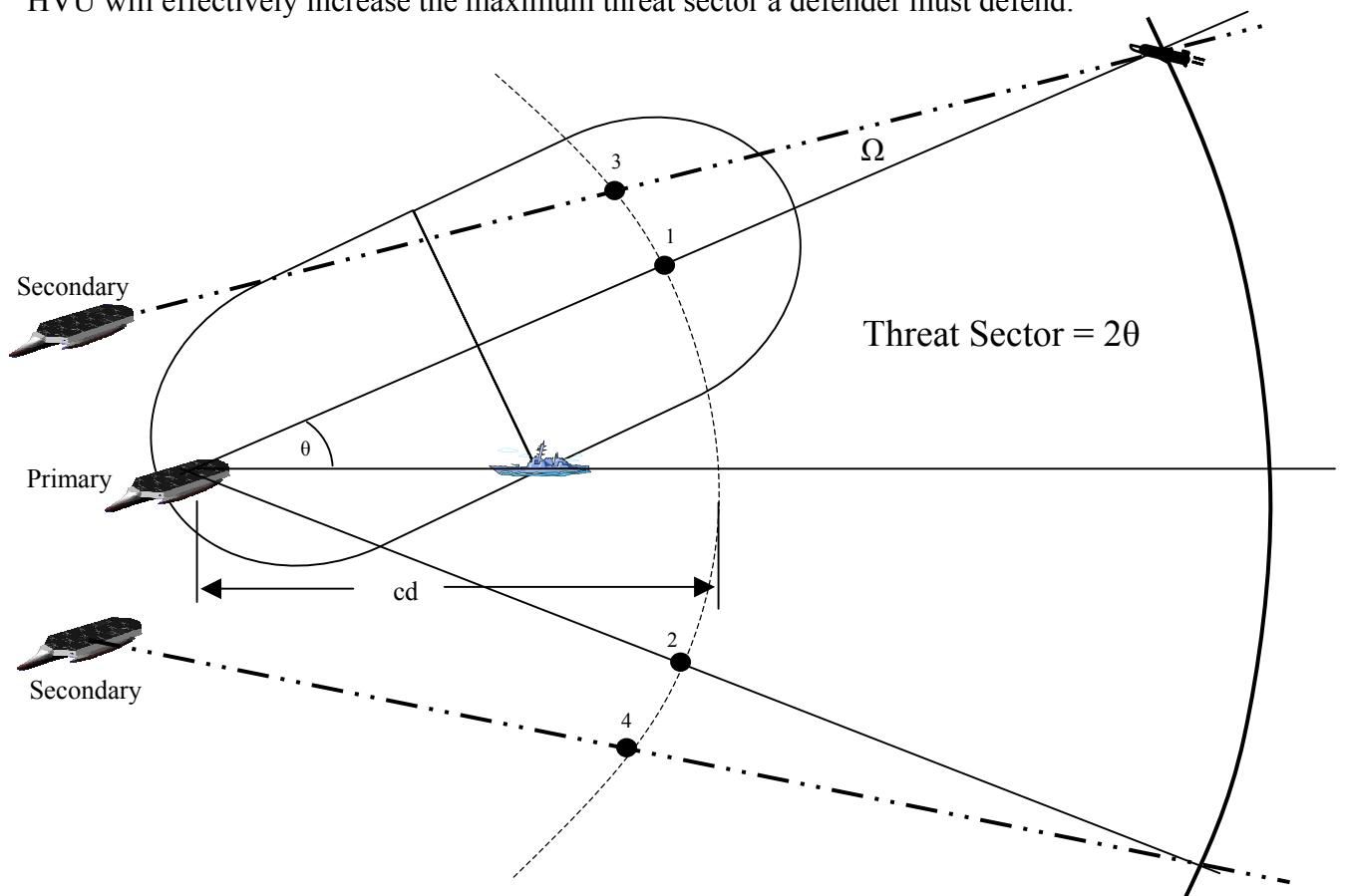


Figure 9 - Multiple HVUs

In the preceding figure, points 1 and 2 are the extreme cpe_{def} a defender must reach in order to engage an attacker with maximum effectiveness within a given threat sector as defined by θ . However, the presence of multiple secondary HVUs laterally offset from the primary can alter the flight profile of an incoming attacker. If the attacker, upon entering the original threat sector, flies a direct path⁴ towards a laterally offset HVU, the attacker will then intersect the defender's cd circle at points 3 or 4. These two new points now define the *effective* threat sector denoted by 2Φ .

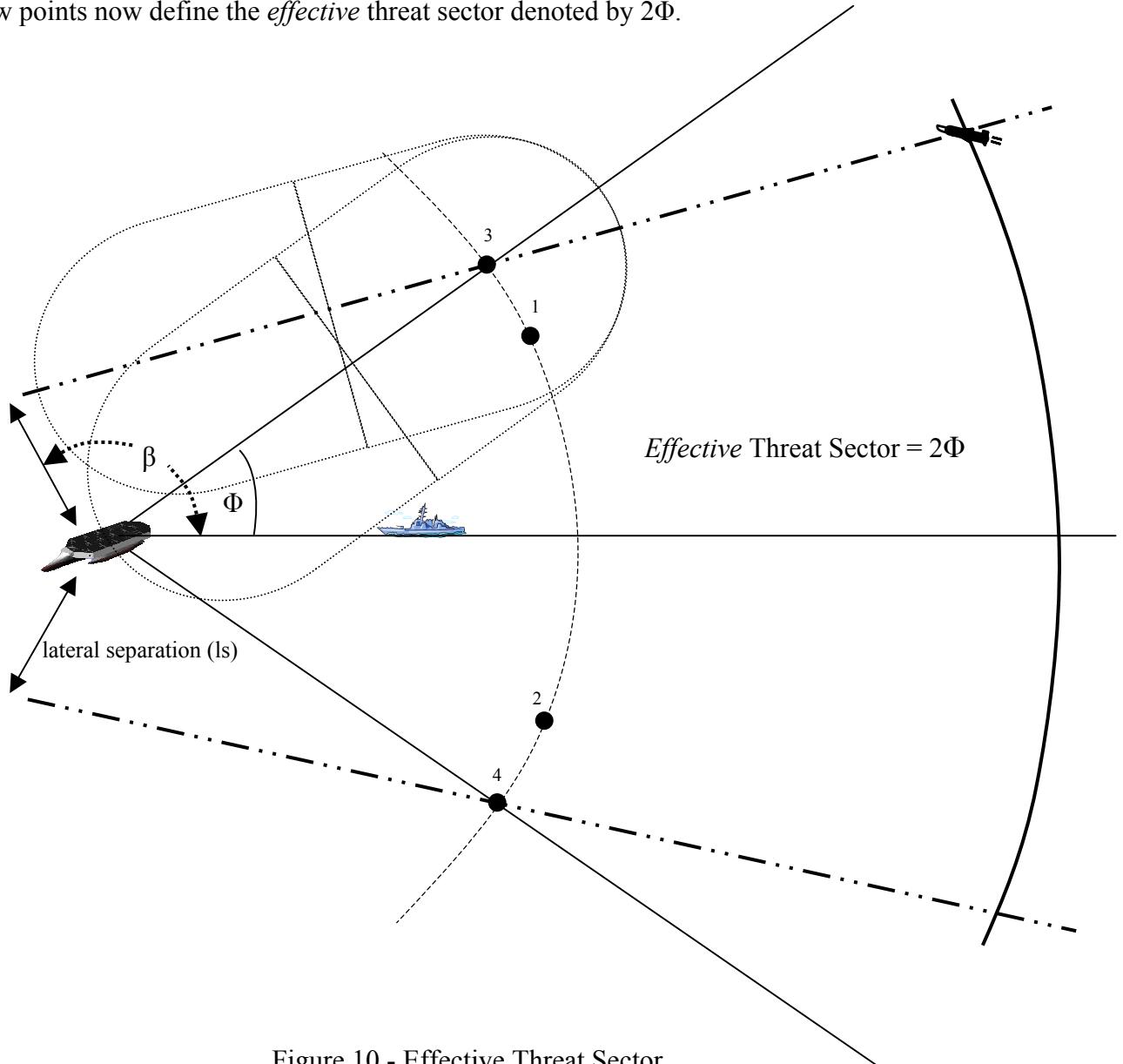


Figure 10 - Effective Threat Sector

⁴ Attackers need not fly direct paths toward HVUs, but due to fuel constraint, are assumed to not deviate greatly from a direct path route.

B. LATERAL SEPARATION (ls)

Let ls be defined as the lateral separation of secondary HVUs from the perspective of an attacker at r_{det} . The lateral separation forms the base of an isosceles triangle with the two equal legs equal to r_{det} . Secondary HVUs can lie on the ray indicated with a dashed line. In order to allow a defender its maximum engagement potential, secondary HVUs should be placed along the ray with angles greater than β . We will now determine the angle Φ and the minimum angle β required to ensure the maximum effective coverage of a laterally separated HVU from the primary HVU by a single defender. The angle β is referenced from the center of the threat sector. First, note the triangle denoted by the bolded lines below and add variable x and angles γ and ω to aid our calculations for Φ and β .

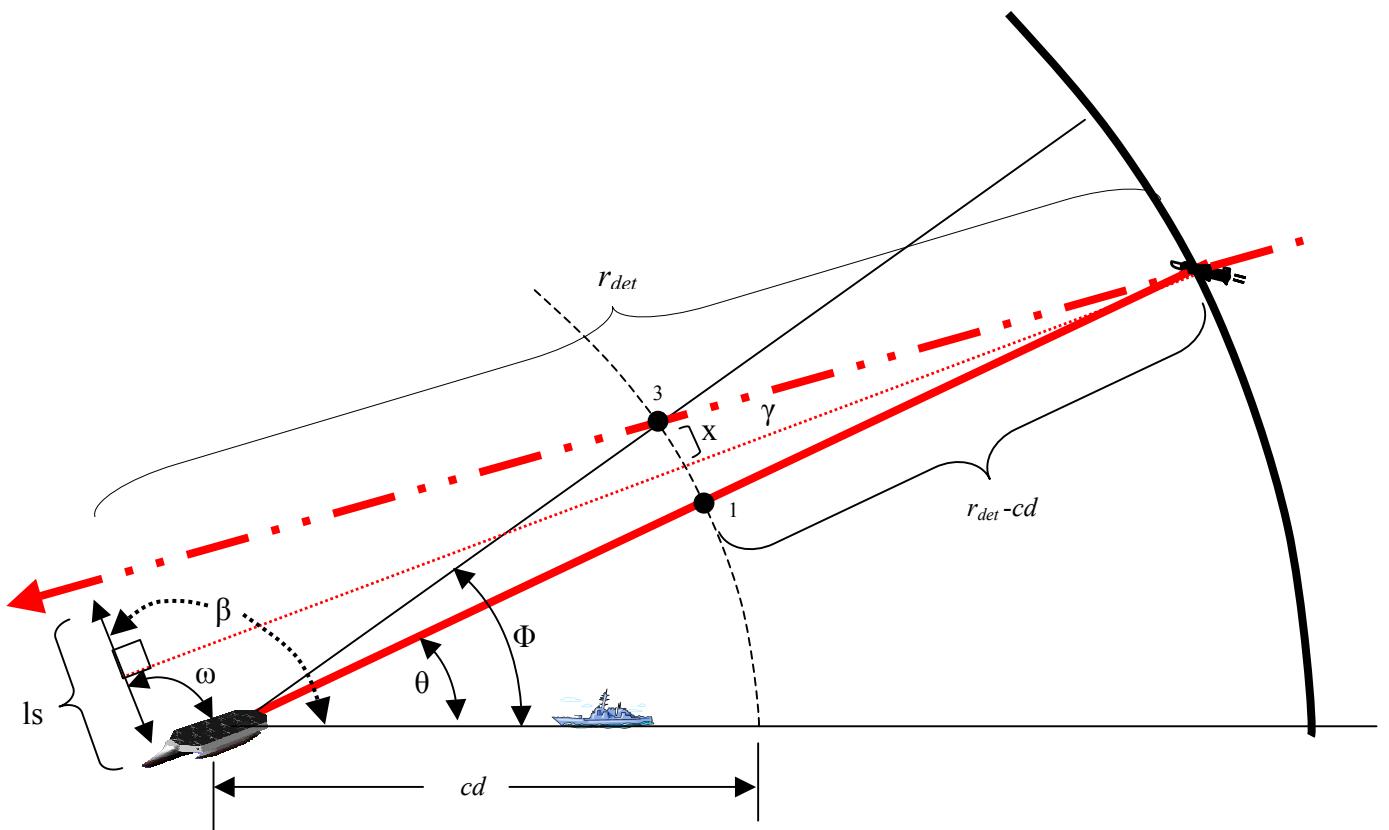


Figure 11 - Effective Threat Sector Calculations I

$$\gamma = \arcsin [(ls / 2) / r_{det}] \quad \text{Eq [22]}$$

$$\omega = 90^\circ - \gamma$$

Eq [23]

$$\beta = \omega + \theta$$

Eq [24]

The variable x is half the distance between points 1 and 3 and can be approximated by the following:

$$x = \sin \gamma \times (r_{det} - cd)$$

Eq [25]

We next utilize the triangle in the following figure denoted by bolded lines. We will add the angle τ to assist in calculations.

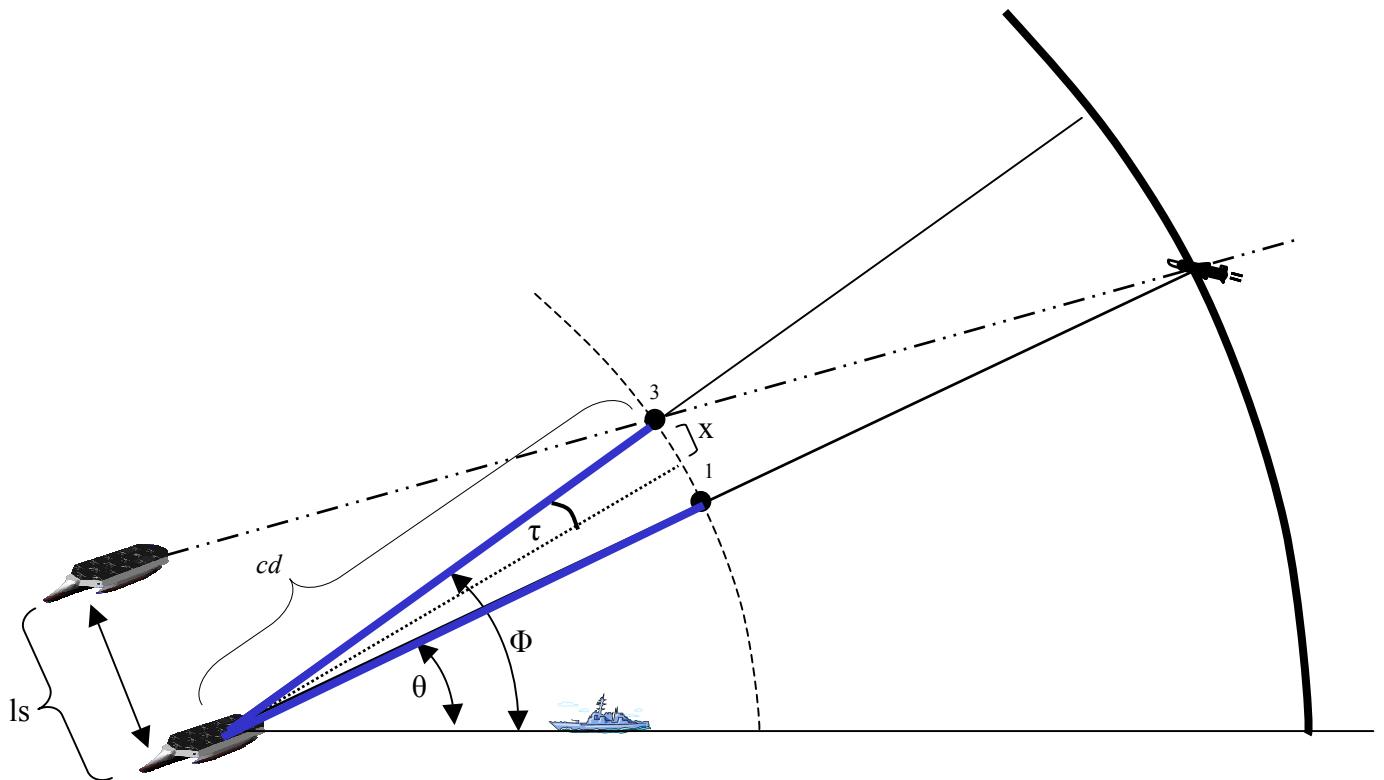


Figure 12 - Effective Threat Sector Calculations II

$$\tau = \arcsin (x / cd)$$

Eq [26]

We can see from the above figure that Φ is the sum of angle θ plus 2τ :

$$\Phi = \theta + 2\tau$$

Eq [27]

Additional HVUs can be protected in this scenario so long as they stay in the *shadow region* as denoted in the following figure.

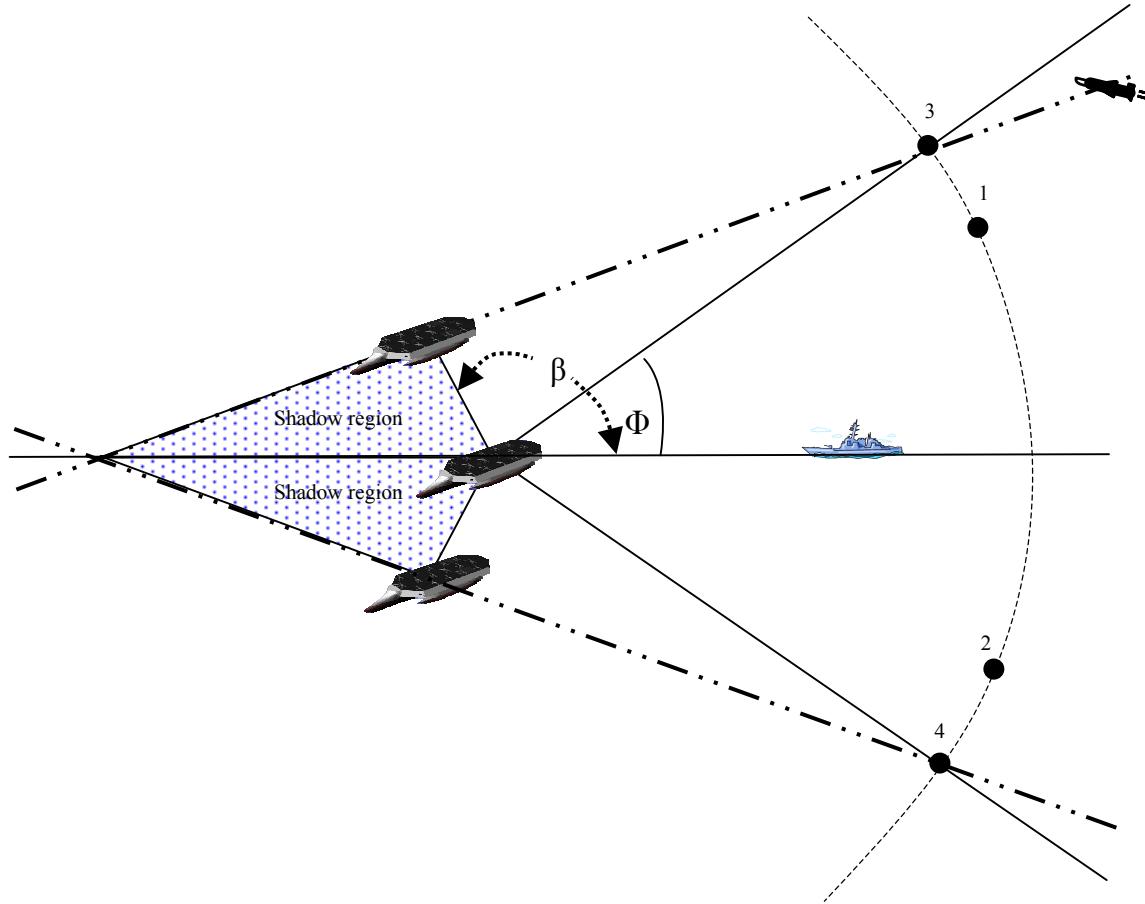


Figure 13 - Shadow Region

In summary, this section determines how lateral separation requirements affect the effective threat sector. It also determines the minimum angle β to allow secondary HVUs to remain under the umbrella of protection provided by a single defender.

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V. MODEL FORMULATION: 1 (HVU) x 1(DEFENDER) x p(ATTACKERS)

This section of the formulation focuses on defending the Sea Base against a High Density Threat (HDT) attack.

Although the physical size of the HDT depends on the attackers' distance from one another and the number of attackers, HDTs, by their very definition, imply multiple attackers within a limited area or volume. Thus, we shall consider HDTs as if co-located.

A. MAXIMUM SALVO NUMBER CALCULATION

Equation [4] is actually an upper bound on P_k as it assumes perfect and instantaneous battle damage assessment (BDA) and instantaneous interception by the defenders weapon system. The US Navy has been researching directed energy weapons in which the assumption of infinite weapon speed would be appropriate. In a more traditional scenario where the defender uses a guided missile system, a defender's fire rate can be affected dramatically when BDA and the defender's weapon time of flight are considered.

Section 1 considers the situation where BDA is instantaneous and focuses on the effects of weapon flight time. Section 2 then includes BDA into the formulation.

1. Effects of Weapon Flight Time

Let us first consider how the defender's weapon time of flight may affect the maximum number of salvos against an incoming attacker. *Naval Operations Analysis* (1999) aptly describes this situation as the SAM/ASM speed and distance relationships.⁵ In this case, SAM (surface-to-air missile) represents the defender's weapon and ASM (anti-ship missile) represents the attacker. To address this relationship, let us define the variables that comprise it.

r_{det} = maximum detection range from the defender

r_{max} = maximum intercept range of defender's weapon system

r_{min} = minimum intercept range of defender's weapon system

⁵ Daniel H. Wagner, *et. al.* ed., *Naval Operations Analysis*, Annapolis, Maryland: Naval Institute Press, 1999, pg. 315-318

v_d = velocity of the defender's weapon system

v_a = velocity of the attacker

$$v_{in} = v_d / (v_d + v_a)$$

$$v_{out} = v_d / (v_d - v_a)$$

t_i = time between i^{th} launch and i^{th} intercept

$$x_1 = \text{first intercept of defender's weapon against an attacker} = \begin{cases} r_{\max} & \text{if inbound} \\ r_{\min} & \text{if outbound} \end{cases}$$

x_i = distance of i^{th} intercept from the defender

i_{total} = total number of salvos against an attacker

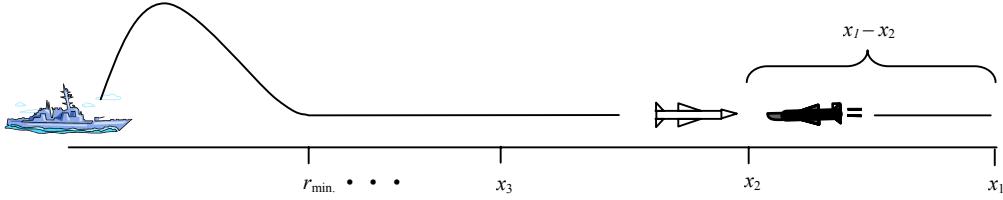


Figure 14 - Salvo Calculations

Assume that an incoming attacker is detected beyond the maximum intercept range ($r_{\text{det}} > r_{\max}$) and that at $x_1 = r_{\max}$ it is intercepted. At this time, the second salvo is launched. Since both the defender's weapon and the attacker have finite velocities, the second intercept will take place at some point x_2 which lies between x_1 and the defender. If the defender's weapon travels to x_2 by time t_2 , it follows that the attacker travels a distance $x_1 - x_2$ during the same time period t_2 which yields the following:

$$v_d t_2 = x_2$$

$$v_a t_2 = x_1 - x_2$$

$$t_2 = \frac{x_2}{v_d} = \frac{x_1 - x_2}{v_a} \quad \text{Eq [28]}$$

Solving for x_2 ,

$$x_2 = r_{\max} \frac{v_d}{v_d + v_a} \quad \text{and let } v_{in} = \frac{v_d}{v_d + v_a}$$

$$\text{Thus, } x_2 = x_1 v_{in} \quad \text{Eq [29]}$$

If the attacker is not killed at x_1 , the defender will fire a third salvo.

$$v_d t_3 = x_3$$

$$v_a t_3 = x_2 - x_3$$

$$t_2 = \frac{x_3}{v_d} = \frac{x_2 - x_3}{v_a}$$

$$x_3 = x_2 \frac{v_d}{v_d + v_a} = x_2 v_{in} \quad \text{Eq [30]}$$

Substituting Eq [29] into Eq [30] yields,

$$x_3 = x_1 v_{in}^2$$

Similar substitutions for subsequent salvos yield,

$$x_i = x_1 v_{in}^{i-1} \quad \text{Eq [31]}$$

If the defender's weapons system has a minimum range (r_{min}), then the intercept point of the i^{th} salvo x_i must be greater or equal to r_{min} . Thus, the maximum number of salvos a defender can fire at an incoming attacker can be found by setting x_i equal to r_{min} .

$$x_i = r_{min} \quad \text{Eq [32]}$$

Substituting Eq [32] into Eq [31] yields,

$$r_{min} = x_1 v_{in}^{i-1} \quad \text{Eq [33]}$$

We now solve for i by use of logarithms,

$$i_{in} = \left\lceil \frac{\ln(r_{min}) - \ln(r_{max})}{\ln(v_{in})} \right\rceil + 1 \quad \text{Eq [34]}$$

Here, $\lfloor \cdot \rfloor$ denotes the floor of the expression since fractional salvos are not possible.

If the attacker's speed is greater than the defender's weapon system's speed, then no further engagements can occur against an outbound attacker. However, if this is not the case, then the defender will be able to fire i_{out} number of salvos. First, to determine i_{out} , let the initial salvo begin at the $x_1 = r_{\min}$ point. At this time, the second salvo is launched. If the defender's weapon travels to x_2 by time t_2 , it follows that the attacker travels a distance $x_1 - r_{\min}$ during the same time period t_2 which yields the following:

$$\begin{aligned} v_d t_2 &= x_2 \\ v_a t_2 &= x_2 - x_1 \\ t_2 &= \frac{x_2}{v_d} = \frac{x_2 - x_1}{v_a} \end{aligned}$$

Eq [35]

Solving for x_2 ,

$$x_2 = x_1 + \frac{v_d}{v_d - v_a} \quad \text{and let } v_{\text{out}} = \frac{v_d}{v_d - v_a}$$

$$\text{Thus, } x_2 = x_1 v_{\text{out}} \quad \text{Eq [36]}$$

Subsequent engagements follow the same sequence as Eq [30] through Eq [33] except for exchanging r_{\max} and v_{in} with r_{\min} and v_{out} . Additionally, the value of r_{\max} in the outbound case depends on the distance between the defender and HVU. If the HVU is far enough away from the defender, the missile's fuel capacity determines r_{\max} . If the HVU and the defender are positioned close to each other, then their separation distance determines r_{\max} .

Let,

d = distance between the defender and HVU

r_{weap} = maximum weapon travel range (fuel dependent)

Thus,

$$r_{\max} = \min(d, r_{\text{weap}})$$

and,

$$i_{\text{out}} = \begin{cases} \left\lfloor \frac{\ln(r_{\max}) - \ln(r_{\min})}{\ln(v_{\text{out}})} \right\rfloor + 1 & \text{if } v_d > v_a \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq [37]}$$

Thus the total number of possible salvos is given by,

$$i_{\text{total}} = i_{\text{in}} + i_{\text{out}} \quad \text{Eq [38]}$$

$$i_{\text{total}} = \left\lfloor \frac{\ln(r_{\min}) - \ln(r_{\max})}{\ln(v_{\text{in}})} \right\rfloor + \left\lfloor \frac{\ln(r_{\max}) - \ln(r_{\min})}{\ln(v_{\text{out}})} \right\rfloor + 2$$

2. Battle Damage Assessment (BDA)

Let us now consider a situation where the defender can uniquely track each attacker that comprises the HDT by radar. The defender can be argued to have a 100% BDA capability since it can determine the success or failure of an engagement by observing a decrease or constancy in the number of attackers in the HDT. However, this BDA will not be instantaneous. A radar system will normally maintain a killed track in its system for a few seconds while it confirms that the track is actually no longer present or just temporarily lost from detection. In addition, the presence of debris may temporary clutter the defender's radar system, displaying several unimportant targets. An attacker can produce debris if engaged successfully by a defender's munitions. Debris can also be produced by the defender's munitions (i.e., high explosive warhead delivered by missile) irrespective of the engagement result. In the presence of debris, the defender will not be able to accurately discern the number of attackers in the HDT from the unimportant targets, thereby preventing the defender from determining the success or failure of the engagement. However, debris will eventually be dropped from the radar system due to impact with the ground or ocean or by computer algorithms that consider velocity vectors and remove unimportant targets. An algorithm may, for example, identify targets as debris that have small velocities and/or are decreasing rapidly in altitude, and remove

them from the system. Thus, only after some BDA time has expired will the radar system be able to resolve the number of attackers in the HDT, and thereby eventually allow a defender to determine the effectiveness of a particular engagement. Let τ denote the time required for BDA. If the defender waits for BDA before conducting another engagement, then the attacker will continue to advance during this time. To explore the effects of BDA on i_{total} , let us again consider the SAM/ASM speed and distance relationship with the addition of three more variables.

- τ = time required for BDA
- y_i = distance from defender at the $i+1$ salvo launch
- δ = distance traveled by the attacker during BDA period

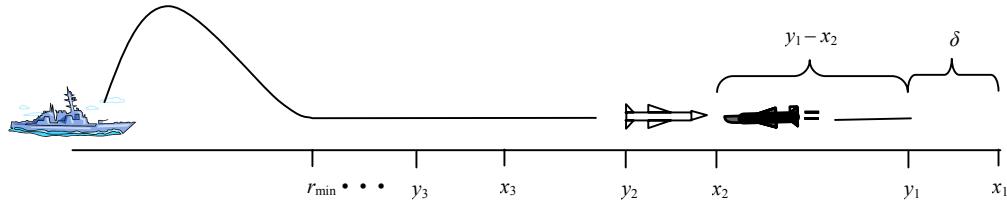


Figure 15 - Salvo Calculations with BDA

Previously, it was assumed that a defender having failed to kill an attacker at x_i , would immediately launch another salvo. Thus, the attacker would travel a distance of $x_n - x_{n+1}$ while the defender's weapon travels to x_{i+1} during the same time period. However, if we consider the time required for BDA, then the attacker will travel a distance of τv_a from x_i before the defender fires. Let δ denote this distance.

$$\delta = \tau v_a$$

Thus, after an intercept at $x_1 = r_{\max}$, the defender will wait τ time units to confirm a success or failure before launching another salvo. If the attacker is killed at x_1 , then no further intercept takes place. If the attacker survives the intercept at x_1 , it will travel

distance δ towards the defender while the defender awaits BDA before a subsequent salvo is launched and is given by the following:

$$y_1 = x_1 - \delta \quad \text{Eq [39]}$$

Thus, Eq [28] is modified to account for BDA.

$$\begin{aligned} v_d t_2 &= x_2 \\ v_a t_2 &= y_1 - x_2 \\ t_2 &= \frac{x_2}{v_d} = \frac{y_1 - x_2}{v_a} \end{aligned} \quad \text{Eq [40]}$$

Solving for x_2 ,

$$x_2 = y_1 \frac{v_d}{v_d + v_a} \text{ and again let } v_{in} = \frac{v_d}{v_d + v_a}$$

$$\text{Thus, } x_2 = y_1 v_{in} \quad \text{Eq [41]}$$

Substituting Eq [39] into Eq [41] yields,

$$x_2 = (x_1 - \delta) v_{in} \quad \text{Eq [42]}$$

If the attacker is not killed at x_2 , the defender will again wait τ time units for BDA before firing a second salvo allowing the attacker to travel another distance of δ .

$$y_2 = x_2 - \delta \quad \text{Eq [43]}$$

$$\begin{aligned} v_d t_3 &= x_3 \\ v_a t_3 &= y_2 - x_3 \\ t_3 &= \frac{x_3}{v_d} = \frac{y_2 - x_3}{v_a} \\ x_3 &= y_2 \frac{v_d}{v_d + v_a} = y_2 v_{in} \end{aligned} \quad \text{Eq [44]}$$

Substituting Eq [42] and Eq [43] into Eq [44] yields,

$$x_3 = (x_1 - \delta)v_{in}^2 - \delta v_{in}$$

Similar substitutions for subsequent salvos yield,

$$x_i = (x_1 - \delta)v_{in}^{i-1} - \delta(v_{in}^1 + v_{in}^2 + \dots + v_{in}^{i-2}) \quad \text{Eq [45]}$$

Since $(v_{in}^1 + v_{in}^2 + \dots + v_{in}^{i-2})$ is a geometric series, it can be expressed as,

$$(v_{in}^1 + v_{in}^2 + \dots + v_{in}^{i-2}) = \frac{1 - v_{in}^{i-1}}{1 - v_{in}} \quad \text{Eq [46]}$$

Substituting Eq [47] into Eq [46] yields,

$$x_i = (x_1 - \delta)v_{in}^{i-1} - \delta \left(\frac{1 - v_{in}^{i-1}}{1 - v_{in}} \right) \quad \text{Eq [47]}$$

Equating x_i to r_{\min} and solving for v_{in}^{i-1} yields,

$$v_{in}^{i-1} = \frac{r_{\min}(1 - v_{in}) + \delta}{(r_{\max} - \delta)(1 - v_{in}) + \delta} \quad \text{Eq [48]}$$

Using logarithms to solve for i yields,

$$i_{in} = \left\lceil \frac{\ln \left(\frac{r_{\min}(1 - v_{in}) + \delta}{(r_{\max} - \delta)(1 - v_{in}) + \delta} \right)}{\ln(v_{in})} \right\rceil + 1 \quad \text{Eq [49]}$$

The outbound case with BDA can be solved in a similar manner as the inbound case with BDA with minor adjustments. As mentioned previously, an outbound case is only feasible if the defender's weapon speed is greater than the attacker's speed. In the outbound case, let the initial intercept occur at $x_1 = r_{\min}$. If the attacker is killed, there are no further intercepts. If the attacker survives, then it will travel distance δ away from the defender while the defender awaits BDA before a subsequent launch and is given by:

$$y_1 = x_1 + \delta \quad \text{Eq [50]}$$

Thus, Eq [35] is modified to account for the outbound case with BDA.

$$v_d t_2 = x_2$$

$$v_a t_2 = x_2 - y_1$$

$$t_2 = \frac{x_2}{v_d} = \frac{x_2 - y_1}{v_a} \quad \text{Eq [51]}$$

Solving for x_2 ,

$$x_2 = y_1 \frac{v_d}{v_d - v_a} \quad \text{and again let } v_{out} = \frac{v_d}{v_d - v_a}$$

Thus,

$$\begin{aligned} x_2 &= y_1 v_{out} \\ x_2 &= (x_1 + \delta) v_{out} \end{aligned} \quad \text{Eq [52]}$$

As in the inbound case with BDA, similar substitutions for subsequent intercepts yield,

$$x_i = (x_1 + \delta) v_{out}^{i-1} + \delta(v_{out}^1 + v_{out}^2 + \dots + v_{out}^{i-2}) \quad \text{Eq [53]}$$

Substituting Eq [46] into Eq [53] yields,

$$x_i = (x_1 + \delta) v_{out}^{i-1} + \delta \left(\frac{1 - v_{out}^{i-1}}{1 - v_{out}} \right) \quad \text{Eq [54]}$$

Equating x_i to r_{\max} and solving for v_{out}^{i-1} yields,

$$v_{out}^{i-1} = \frac{r_{\max} (1 - v_{out}) - \delta}{(r_{\min} + \delta)(1 - v_{out}) - \delta} \quad \text{Eq [55]}$$

Using logarithms to solve for i yields,

$$i_{out} = \begin{cases} \left\lfloor \frac{\ln\left(\frac{r_{max}(1-v_{out})-\delta}{(r_{min}+\delta)(1-v_{out})-\delta}\right)}{\ln(v_{out})} \right\rfloor + 1 & \text{if } v_d > v_a \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq [56]}$$

Thus i_{total} with BDA is given by,

$$i_{total} = \left\lfloor \frac{\ln\left(\frac{r_{min}(1-v_{in})+\delta}{(r_{max}-\delta)(1-v_{in})+\delta}\right)}{\ln(v_{in})} \right\rfloor + \left\lfloor \frac{\ln\left(\frac{r_{max}(1-v_{out})-\delta}{(r_{min}+\delta)(1-v_{out})-\delta}\right)}{\ln(v_{out})} \right\rfloor + 2 \quad \text{Eq [57]}$$

B. HDT PROBABILITY OF KILL WITH MARKOV CHAINS

If a single shot probability of kill is given as p , then the probability that the attacker survives is $1-p$. Assuming independence, if there are two attackers and two rounds in a salvo with each directed at a different attacker, then the probability that both attackers survive the salvo is $(1-p) * (1-p)$. The probability that only one is killed is the probability the first attacker is killed and the second attacker survives, plus the probability the first attacker survives and the second attacker is killed: $[(1-p) * p] + [p * (1-p)]$. Accordingly, the probability that both attackers are killed is $p * p$. A Markov chain whose state is the number of surviving attackers (T) results from these observations with transition matrix A as shown in Figure 16.

Let T = alive attackers

Salvo size = 2

		Alive Attackers after Salvo					
		5T	4T	3T	2T	1T	0T
Initial Number of Attacker before Salvo	5T	$(1-p)^2$	$2[(1-p)p]$	p^2	0	0	0
	4T	0	$(1-p)^2$	$2[(1-p)p]$	p^2	0	0
	3T	0	0	$(1-p)^2$	$2[(1-p)p]$	p^2	0
	2T	0	0	0	$(1-p)^2$	$2[(1-p)p]$	p^2
	1T	0	0	0	0	$(1-p)$	p
	0T	0	0	0	0	0	0

Figure 16 – Transition Matrix A for Salvo Calculations

In Figure 16, each horizontal row shows the transition possibilities. Zeroes in the cell denote the inability to transition into the column from the row. For example, if the state is 5T, and the defender has only a two round salvo capability, the probability of transitioning from 5T to 2T, 1T, or 0T is zero. Similarly, if the initial state is 3T, then the probability of transitioning to 4T or 5T is zero since we assume that killed attackers cannot come back alive. In addition, it is assumed in this model that the defender launches its maximum round for every salvo, except only as many rounds as attackers if there are fewer attackers than the defender's maximum capacity. For example, a defender will launch a 2 missile salvo so long as there are at least 2 attackers. If there is only 1 attacker, then only one missile will be launched. This parameter can be adjusted for different tactics. To calculate the state transition distribution after n salvos, take VA^n where A is the state transition matrix and V is $[1, 0, 0, \dots]$. This distribution can then be compared to operational requirements for adequacy. Thus, the model can provide the commander the probability of successfully killing a certain number of attackers after n salvos. In addition, the model can provide the number of salvos required in order to achieve a certain probability of success.

C. TYPE I AND TYPE II ERRORS

Let us define a Type I error as when a defender believes an attacker is alive but is actually killed. A Type II error is when a defender believes an attacker is killed but is actually alive. In battles between ships, such errors are not uncommon. A reason may be

because a “killed” ship may not sink instantly if at all. Subsequently, a damaged ship can give the appearance of being “killed”, but still be combat capable. For example, during the Battle of Coral Sea, USS LEXINGTON and YORKTOWN took heavy torpedo and bomb hits. The Japanese reported both as sunk when actually the YORKTOWN received only light damage and the YORKTOWN maintained its 25 knot steaming capability and continued flight operations.⁶ In a modern air warfare scenario, it is difficult for Type I and II errors to be made. BDA for an air target can be made either visually or by radar. For visual BDA, an attacker can be evaluated as “killed” when it is observed to have made contact with a missile or other munitions and explodes. It can also be observed as “killed” if after contact with a munition, it loses control and impacts the ground or sea. Similarly, an air target can be visually evaluated as “alive” if the munition is observed to have missed, or the air target continues to fly in a controlled manner. For BDA with radar, an air target can be evaluated as “killed” if it no longer returns electromagnetic (EM) pulse either due to explosion or impact with the ground or sea. In either case, Type II error is difficult since air targets that have exploded or impacted the surface will normally not be evaluated as “alive”. Similarly, Type I error is difficult since an “alive” air target that is still flying cannot give the appearance of exploding or impacting the surface without actually doing so.⁷ Thus, for the analyses in this model, we will assume no Type I or II errors.

D. MISSILE FIRING POLICIES AGAINST HDTs

HDTs add another layer of complication to defense at sea. The factors that go into successfully killing a small, incoming raid of attackers include such things as single-shot probability of kill, probability of detection, detection time or range, velocity vectors, and the number of shots. The number of shots can be driven by a weapon’s maximum and minimum ranges, and velocity vectors, as described in Eq [57]. As the number of attackers that comprise the HDT grows larger, the total number of engagements plays an increasingly important role in determining the level of success a defender may have against the attackers. With HDTs, if the number of attackers exceeds the number of

⁶ Ronald H. Spector, *Eagle Against the Sun*, New York: The Free Press, 1985, pg. 162

⁷ Stealth aircraft may be able to hide from radar, however, this capability is constant and cannot be turned on and off to give a defender’s radar the appearance of it having exploded or crashed.

engagements, then the probability of successfully killing the entire raid is zero. For example, if a HDT is comprised of 10 attackers and there are only 4 opportunities to fire a one round salvo, a defender can at most kill 4 attackers. If a defender can fire a two round salvo as current capabilities allow, and each missile is directed at a different attacker, a defender can at most kill 8 attackers. If each salvo is directed at a single attacker, then again, at most only 4 attackers can be killed. Therefore, this section will explore strategies a defender might employ to increase the number of engagements against HDTs.

One method might be to increase the fire rate of the defender or the salvo size. However, some ships already have missile fire intervals of only a few seconds. The Mk-13 Guided Missile Launching System (GMLS) on current US Navy frigates has a continuous fire interval of 8.09 seconds.⁸ As for salvo size, with the advent of the Vertical Launch System (VLS), each missile is already in its launch canister poised for near simultaneous firing⁹, easily accommodating potentially large salvo sizes. However, current fire rate and salvo size tend to be more a function of the number of available fire control (FC) radars. In other words, a defender's salvo size can be limited by the number of fire control (FC) radars it possesses. One reason is that some EM seeking missiles require continuous illumination from launch to intercept. If a ship with three FC radars launches a three-missile salvo, a subsequent salvo cannot be launched until the FC radars have completed guiding the previous salvo to termination.

The Aegis combat system, however, can be argued to have overcome this limitation since it uses a search radar (SPY-1) to provide mid-course guidance to a missile. Thus, a FC radar is only required to illuminate an attacker at the terminal phase of the intercept, allowing more missiles than FC radars to be in flight. This would allow a ship equipped with the Aegis combat system to better utilize its faster rate of fire since a missile launch does not require a free FC radar. However, this capability can be rendered ineffective if SPY-1 tracking resolution is not high. If the attackers in the HDT have small radar cross sections (RCS) and are relatively close to one another, SPY-1 may not be able to completely resolve one or more attackers as individual tracks. Currently, the

⁸ <http://www.fas.org/man/dod-101/sys/ship/weaps/mk13-gmls.pdf>, 19NOV03

⁹ <http://www.fas.org/man/dod-101/sys/ship/weaps/an-spy-1.htm>, 19NOV03

SPY-1 weapon control system (WCS) does not allow subsequent salvos against a specific target track until it has been evaluated as killed.¹⁰ Thus, in the worst case where SPY-1 sees a HDT as a single target, the system behaves as if under the requirement for continuous illumination, thereby countering SPY-1's multiple in-flight salvo capability. Although it may still be possible for a defender to fire a large salvo against this aggregated HDT, because of the narrow beam of the FC radar, the salvo will tend to converge on the single attacker that happens to be illuminated in the group. This may increase the probability of kill for that single target, but will have minimal effect on the HDT as a whole.

1. Broad vs. Narrow Beam FC Radar

Given the situation where a HDT is seen as a single target, a defender may want to explore an alternative means of maximizing the potential number of engagements. One such alternative may be to modify FC radars to accommodate a wider beam.

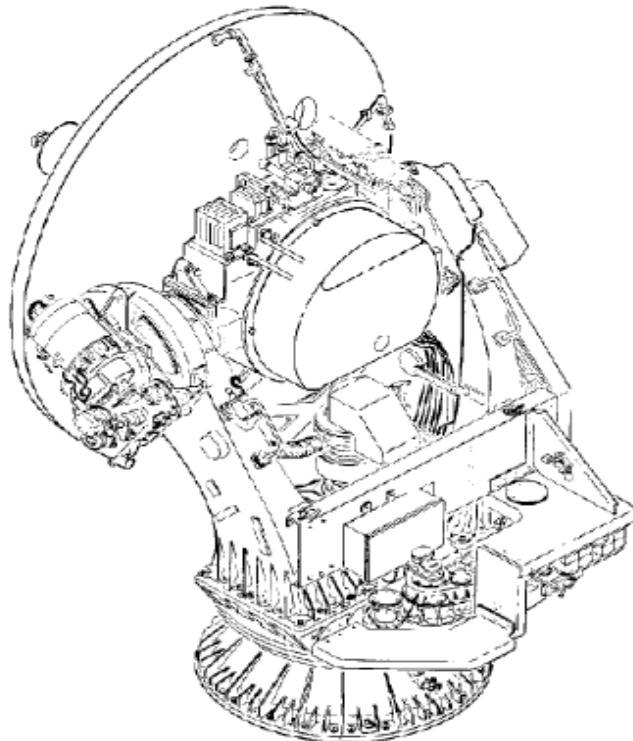


Figure 17 - Mk 92 Fire Control System¹¹

¹⁰ <http://www.fas.org/man/dod-101/sys/ship/weaps/an-spy-1.htm>, 19NOV03

¹¹ <http://www.fas.org/man/dod-101/sys/ship/weaps/an-spy-1.htm>, 19NOV03

Currently, FC radars are used to focus narrow beams of electromagnetic (EM) energy at a single target for illumination. Illumination is the process by which high frequency EM energy is reflected off a target. An EM seeking missile then detects this reflected energy and homes in on that target. The FC energy beams are narrow to ensure that other objects that may be nearby, such as friendly aircraft, are not unintentionally illuminated and homed in on by the defender's missile. (See Figure 18)

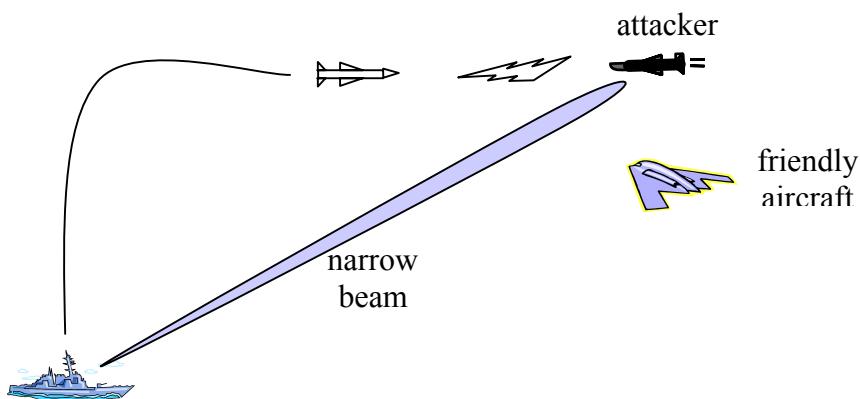


Figure 18 - Narrow Beam Engagement

2. Narrow Beam Model

Using FC radars with a narrow beam, the defender will direct its FC radar at the HDT. The narrow beam FC radar will choose only one attacker. The defender then fires a salvo at this attacker, waits for intercept, and then moves on to fire at the next randomly chosen attacker without waiting for BDA. The defender will continue to fire at the HDT in this manner, of course skipping over attackers that have been killed and dropped from the system, until the entire HDT is killed. The defender must decide how many missiles to shoot on each salvo depending on:

- (1) remaining salvos,
- (2) remaining missiles, and
- (3) remaining attackers.

Let,

p = single shot probability of kill of a defender's weapon

$q = 1 - p$

s = number of remaining salvos

a = number of attackers remaining before a salvo

y = number of attackers remaining after a salvo

x = number of weapons fired in a salvo

w = number of remaining weapons in inventory

$EV_N(s, w, a)$ = maximum expected number of attackers killed if there are s remaining salvos, w remaining weapons, and a attackers (Narrow Beam)

$P_N(a, x, y)$ = probability of having y remaining attackers after firing x missiles at a attackers in a single salvo (Narrow Beam)

If a defender decides to fire x missiles at a attackers s salvos remaining, then the defender will have, $w - x$ weapons, either y or $y - 1$ attackers, and $s - 1$ salvos remaining.

This process is continued until either salvos, weapons or alive attackers are exhausted. This process is then repeated for all possible values of x, y and s .

To find the average number of attackers killed after s salvos,

$$EV_N(s, w, a) = \max_{(0 \leq x \leq w)} \left\{ \sum_{y=a}^a P_N(a, x, y) [(a-y) + EV_N(s-1, w-x, y)] \right\} \quad \text{Eq [58]}$$

$$\text{Where } P_N(a, x, y) = \begin{cases} (1-p)^x & \text{if } y = a \\ 1 - (1-p)^x & \text{if } y = a-1 \end{cases} \quad \text{Eq [59]}$$

The above algorithm is recursive and is explored through use of a computer. The optimal distribution of missiles is stored in an array.¹² An extended example is given in section 4 below.

The disadvantage of a narrow beam FC radar is that the number of attackers that can be fired on can be limited by the number of salvos. If the number of attackers in the

¹² This algorithm was implemented in Java and can be found in Appendix A.

HDT is larger than the number of salvos, a defender will not be able to fire at all attackers in the HDT.

3. Wide Beam Model

Instead of a narrow beam illuminating just a single attacker, conceptually a wide beam can be used to illuminate the multiple attackers in the HDT.

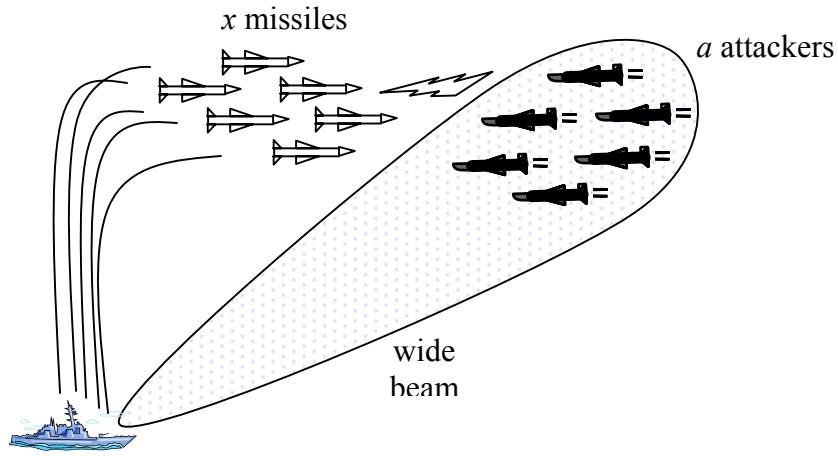


Figure 19 - Wide Beam Engagement

This model is similar to the Narrow Beam model except, since multiple attackers are illuminated, a large salvo will not tend to converge on just a single attacker. Let us assume that each missile chooses an attacker uniformly, randomly, and independently of the others. The missile then kills its chosen attacker with probability p^{13} . Firing in this manner carries the risk of waste, since some missiles may choose attackers already killed by other missiles, but it also permits large salvo sizes to be spread over multiple attackers at each salvo.

First let us calculate the probability distribution for having y remaining attackers in a salvo size of x missiles directed against a attackers.

¹³ This situation can be described as the Urn model. Kress, Moshe, *Class Notes, Advanced Combat Models*, Operations Research Department, Naval Postgraduate School, Monterey, California, March, 2003.

Let,

$EV_W(s, w, a)$ = maximum expected number of attackers killed if there are s remaining salvos, w remaining weapons, and a attackers (Wide Beam)

$P_W(a, x, y)$ = probability of having y remaining attackers after firing x missiles at a attackers in a single salvo (Wide Beam)
(other variables as defined for the Narrow Beam Model)

If $0 \leq y \leq a$, we have Eq [60]:

$$P_W(a, x, y) = \sum_{j=0}^x \underbrace{\binom{x}{j} \left(\frac{1}{a}\right)^j \left(1 - \frac{1}{a}\right)^{x-j}}_{(1)} \left[\underbrace{q^j P_W(a-1, x-j, y-1)}_{(2)} + \underbrace{(1-q^j) P_W(a-1, x-j, y)}_{(3)} \right]$$

Where,

(1) is the binomial probability that j out of x missiles randomly choose the last attacker.

(2) q^j is the probability that the last attacker survives the j missiles, in which case there must be $y-1$ survivors among the $a-1$ attackers that are fired on by $x-j$ missiles.

(3) $(1 - q^j)$ is the probability that the last attacker is killed by the j missiles, in which case there must be y survivors among the $a-1$ attackers that are fired on by $x-j$ missiles.

Now that we have the probability distribution given the number of remaining salvos, missiles, and attackers, we can now evaluate the expected number of attackers killed (EV) by substituting this probability distribution (Eq [60]) into Eq [58] and exploring all possible combinations of salvos, missiles, and attackers.

$$EV_W(s, w, a) = \max_{0 \leq x \leq w} \sum_{y=0}^a P_W(a, x, y) [(a-y) + EV_W(s-1, w-x, y)] \quad \text{Eq [61]}$$

4. Narrow vs. Wide Beam Model Comparison

In this section, a comparison of the two models is conducted by utilizing the algorithm for EV (Eq [58] and Eq [61]). Figures 20-27 list the recommended salvo size x [s, w, a], given the remaining salvos, weapons, and attackers. The left vertical column indicates the weapons remaining and top horizontal row indicates the number of remaining attackers.

a. Narrow Beam Model Example

Let $s = 4, w = 13, a = 10, p = 0.75$

		y										
		0	1	2	3	4	5	6	7	8	9	10
		1	1	1	1	1	1	1	1	1	1	1
		2	2	1	1	1	1	1	1	1	1	1
		3	3	1	1	1	1	1	1	1	1	1
		4	4	1	1	1	1	1	1	1	1	1
		5	5	2	2	2	2	2	2	2	2	2
w	6	6	6	2	2	2	2	2	2	2	2	2
	7	7	7	2	2	2	2	2	2	2	2	2
	8	8	8	2	2	2	2	2	2	2	2	2
	9	9	9	3	3	3	3	3	3	3	3	3
	10	10	10	3	3	3	3	3	3	3	3	3
	11	11	11	3	3	3	3	3	3	3	3	3
	12	12	12	3	3	3	3	3	3	3	3	3
	13	13	13	4	4	4	4	4	4	4	4	4

Figure 20 – Recommended Salvo Size (x) (Narrow Beam)

The recommended number of missiles to fire in this first salvo is 4, which will all converge on a single randomly chosen attacker.

$s = 3$		y										
		0	1	2	3	4	5	6	7	8	9	10
w	1	1	1	1	1	1	1	1	1	1	1	1
	2	2	1	1	1	1	1	1	1	1	1	1
	3	3	1	1	1	1	1	1	1	1	1	1
	4	4	2	2	2	2	2	2	2	2	2	2
	5	5	2	2	2	2	2	2	2	2	2	2
	6	6	2	2	2	2	2	2	2	2	2	2
	7	7	3	3	3	3	3	3	3	3	3	3
	8	8	3	3	3	3	3	3	3	3	3	3
	9	9	3	3	3	3	3	3	3	3	3	3
	10											

Figure 21 – Recommended Salvo Size (x) (Narrow Beam)

In the second salvo, the number of remaining attackers can vary from 9 to 10 depending on whether the previously selected attacker was killed. The recommended number of missiles to fire in this salvo is 3 in both cases, which will again converge on a single randomly chosen attacker.

$s = 2$		y										
		0	1	2	3	4	5	6	7	8	9	10
w	1	1	1	1	1	1	1	1	1	1	1	1
	2	2	1	1	1	1	1	1	1	1	1	1
	3	3	2	2	2	2	2	2	2	2	2	2
	4	4	2	2	2	2	2	2	2	2	2	2
	5	5	3	3	3	3	3	3	3	3	3	3
	6	6	3	3	3	3	3	3	3	3	3	3
	7											

Figure 22 – Recommended Salvo Size (x) (Narrow Beam)

$s = 1$		y										
		0	1	2	3	4	5	6	7	8	9	10
w	1	1	1	1	1	1	1	1	1	1	1	1
	2	2	2	2	2	2	2	2	2	2	2	2
	3	3	3	3	3	3	3	3	3	3	3	3

Figure 23 – Recommended Salvo Size (x) (Narrow Beam)

Since this is the last salvo, the defender fires whatever is remaining in its inventory to achieve the largest expected number of attackers killed.

Using Eq [61], the expected number of killed attackers is,

$$EV_N = 3.95$$

Generally, a defender implementing the Narrow Beam model will attempt to evenly distribute its weapons inventory over all salvos.

b. Wide Beam Model Example

Let $s = 4, w = 13, a = 10, p = 0.75$

		y										
		0	1	2	3	4	5	6	7	8	9	10
		1	1	1	1	1	1	1	1	1	1	1
		2	2	1	1	1	1	1	1	1	1	1
		3	3	1	1	1	1	1	1	1	1	1
		4	4	1	1	1	1	1	1	1	1	1
		5	5	2	2	2	2	2	2	2	2	2
		6	6	2	2	2	2	2	2	2	2	2
		7	7	2	2	2	2	2	2	2	2	2
		8	8	2	2	2	3	3	3	3	2	2
		9	9	3	3	3	3	3	3	3	3	3
		10	10	3	3	3	3	3	3	3	3	3
		11	11	3	3	3	4	4	4	4	4	3
		12	12	3	3	3	4	4	4	4	4	4
		13	13	4	4	4	4	4	4	4	4	4

Figure 24 - Recommended Salvo Size (x) (Wide Beam)

The recommended number of missiles to fire in this first salvo is 4.

		y										
		0	1	2	3	4	5	6	7	8	9	10
		1	1	1	1	1	1	1	1	1	1	1
		2	2	1	1	1	1	1	1	1	1	1
		3	3	1	1	1	1	1	1	1	1	1
		4	4	2	2	2	2	2	2	2	2	2
		5	5	2	2	2	2	2	2	2	2	2
		6	6	2	2	2	3	2	2	2	2	2
		7	7	3	3	3	3	3	3	3	3	3
		8	8	3	3	3	3	3	3	3	3	3
		9	9	3	3	4	4	4	4	4	4	3

Figure 25 - Recommended Salvo Size (x) (Wide Beam)

The recommended number of missiles to fire in the second salvo is 3 or 4 depending on the number of attackers that were killed in the preceding salvos.

		y										
		$s = 2$										
		0	1	2	3	4	5	6	7	8	9	10
w	1	1	1	1	1	1	1	1	1	1	1	1
	2	2	1	1	1	1	1	1	1	1	1	1
	3	3	2	2	2	2	2	2	2	2	2	2
	4	4	2	2	2	2	2	2	2	2	2	2
	5	5	3	3	3	3	3	3	3	3	3	3
	6	6	3	4	4	3	3	3	3	3	3	3

Figure 26 - Recommended Salvo Size (x) (Wide Beam)

The recommended number of missiles to fire in the third salvo is 3.

		y										
		$s = 1$										
		0	1	2	3	4	5	6	7	8	9	10
w	1	1	1	1	1	1	1	1	1	1	1	1
	2	2	2	2	2	2	2	2	2	2	2	2
	3	3	3	3	3	3	3	3	3	3	3	3

Figure 27 - Recommended Salvo Size (x) (Wide Beam)

As in the Narrow Beam model, the defender fires whatever is remaining in its inventory in the last salvo so long as there are alive attackers.

Using Eq [61], the expected number of killed attackers is,

$$EV_W = 8.38$$

Thus, under the condition when $s < w$, $s < a$, $x_g = s$, and $x_u = w$, the Wide Beam model will always outperform the Narrow Beam model. The compelling reason for using the Wide Beam model in this situation is due to the possibility for the defender to fire each salvo at multiple attackers, while only one attacker can be engaged in the Narrow Beam model. When $s \geq w$, the two models perform equivalently. In this case, both models recommend firing only one missile per salvo to ensure that there are no redundant kills.

VI. NUMERICAL EXAMPLES

This section will provide numerical examples of the concepts developed in the previous sections to illustrate possible insights this model may provide to commanders and system designers in Sea Base Defense.

A. DETERMINING MAX THREAT SECTOR

EXAMPLE 1: To determine the maximum threat sector and number of defenders required.

Given:

Attacker speed	500 kts
Defender speed	30 kts
Detection range	200 nm
Weapons range	30 nm

Using Eq [6]:

$$\begin{aligned} v_{rel} &= v_{tar} - v_{def} \\ &= 500 \text{ kts} - 30 \text{ kts} \\ &= 470 \text{ kts} \end{aligned}$$

Using Eq [7]:

$$\begin{aligned} t_{max} &= 2 \times r_{weap} / v_{rel} \\ &= 2 \times 30 \text{ nm} / 470 \text{ kts} \\ &= 0.1277 \text{ hrs} \end{aligned}$$

Using Eq [11]:

$$\begin{aligned} cd &= r_{weap} [1 + 2 / (\mu - 1)] \quad (\mu = v_{tar} / v_{def}) \\ &= 30 \text{ nm} [1 + 2 / ((500 \text{ kts} / 30 \text{ kts}) - 1)] \\ &= 33.83 \text{ nm} \end{aligned}$$

Using Eq [12]:

$$\begin{aligned} atd &= r_{det} - (cd + r_{weap}) \\ &= 200 \text{ nm} - (33.83 \text{ nm} + 30 \text{ nm}) \\ &= 136.17 \text{ nm} \end{aligned}$$

Using Eq [13]:

$$\begin{aligned}
 t_0 &= atd / v_{tar} \\
 &= 136.17\text{nm} / 500\text{kts} \\
 &= 0.2723 \text{ hrs}
 \end{aligned}$$

Using Eq [14]:

$$\begin{aligned}
 dtd &= t_0 \times v_{def} \\
 &= 0.2723 \text{ hrs} \times 30\text{kts} \\
 &= 8.17\text{nm}
 \end{aligned}$$

Using Eq [20]:

$$\begin{aligned}
 \theta_{\max} &= \sin^{-1} (dtd / cd) \\
 &= \sin^{-1} (8.17\text{nm} / 33.83\text{nm}) \\
 &= 13.98^\circ
 \end{aligned}$$

In this case, solving for θ_{\max} yields 13.98° or a threat sector coverage of approximately 28° . From this, we can now broaden our model to include n defenders. By dividing 360° by the maximum threat sector coverage, the minimum number of defenders required for all around coverage without overlap is approximately 13. The case for overlap is discussed further in the section *Model Limitations and Areas for Further Analysis*.

B. DETERMINING PROBABILITY OF KILL (SINGLE ATTACKER)

EXAMPLE 2: To determine the probability of kill

Given:

Attacker speed	500 kts
Defender speed	30 kts
Detection range	200 nm
Weapons range	30 nm
λ	50 shots/hr
p	0.4

Using Eq [6]:

$$\begin{aligned}
 t_{max} &= 2 \times r_{weap} / v_{rel} \\
 (\text{from example 1}) &= 2 \times 30\text{nm} / 470\text{kts} \\
 &= 0.1277 \text{ hrs}
 \end{aligned}$$

Using Eq [8]:

$$\begin{aligned}
 n &= \lambda \times t_{max} \\
 &= 50 \text{ shots/hr} \times 0.1277 \text{ hrs} \\
 &= 6.38 \text{ salvos}
 \end{aligned}$$

Using Eq [4]:

$$\begin{aligned}
 P_k &= 1 - e^{-\lambda p t} \\
 &= [1 - e^{-50*(0.4)*(0.1277)}] \\
 &= 0.9222
 \end{aligned}$$

Hence, if the defender has a single shot probability of kill of 40%, then the probability of an attacker being killed during the engagement is approximately 92%.

1. Effect of Kill Probability on Threat Sector

EXAMPLE 3: To determine the effect of probability of kill on maximum threat sector.

Building from Example 2, if the desired probability of kill after the completion of the entire engagement is 80%, then the number of salvos can be calculated by solving for n.

Using Eq [4]:

$$\begin{aligned}
 P_k &= 1 - e^{-\lambda p t} \\
 e^{-\lambda p t} &= 1 - P_k \\
 t &= -\log(1 - P_k) / \lambda p \\
 &= -\log(1 - 0.80) / (50 * 0.4) \\
 &= 0.08 \text{ hrs}
 \end{aligned}$$

Using Eq [10]:

$$\begin{aligned}
 ad &= v_{def} \times t \\
 &= 30 \text{ kts} \times 0.08 \text{ hrs} \\
 &= 2.4 \text{ nm}
 \end{aligned}$$

Using Eq [9]:

$$\begin{aligned} cd &= r_{weap} + ad \\ &= 30\text{nm} + 2.4\text{nm} \\ &= 32.4\text{nm} \end{aligned}$$

Using Eq [12]:

$$\begin{aligned} r_{engage} &= cd + r_{weap} \\ atd &= r_{det} - r_{engage} \\ &= 200\text{nm} - (32.4\text{nm} + 30\text{nm}) \\ &= 137.6\text{nm} \end{aligned}$$

Using Eq [13]:

$$\begin{aligned} t_0 &= atd / v_{tar} \\ &= 137.6\text{nm} / 500\text{kts} \\ &= 0.2752 \text{ hrs} \end{aligned}$$

Using Eq [14]:

$$\begin{aligned} dtd &= t_0 \times v_{def} \\ &= 0.2752 \text{ hrs} \times 30\text{kts} \\ &= 8.256\text{nm} \end{aligned}$$

Using Eq [20]:

$$\begin{aligned} \theta_{\max} &= \sin^{-1} (dtd / cd) \\ &= \sin^{-1} (8.256\text{nm} / 32.4\text{nm}) \\ &= 14.76^\circ \end{aligned}$$

Reducing our probability of kill requirement from 95% to 80% shortened our cd and allowed larger flexibility in threat sector coverage by a single defender from 28° to about 30° ($2 \times 14.76^\circ$). Thus, the model provided some insight into the effect of changing the probability of kill and in this scenario showed that relatively large changes in P_k requirements do not affect a single defender's maximum threat sector coverage greatly.

C. EFFECT OF DETECTION RANGE

EXAMPLE 4: To determine the effect of detection range on maximum threat sector

The effect of degradation of radar range due to combat damage or atmospherics can potentially reduce the maximum threat sector coverage by a single defender. In the same light, newer and more robust radar systems that increase radar detection range can increase the maximum threat sector coverage. For example, let the new radar detection range (r_{det}) be 250nm vice 200nm and the other parameters the same as in Example 1.

Using Eq [12]:

$$\begin{aligned} r_{engage} &= cd + r_{weap} \\ atd &= r_{det} - r_{engage} \\ &= 250\text{nm} - (33.83\text{nm} + 30\text{nm}) \\ &= 186.17\text{nm} \end{aligned}$$

Using Eq [13]:

$$\begin{aligned} t_0 &= atd / v_{tar} \\ &= 186.17\text{nm} / 500\text{kts} \\ &= 0.3723 \text{ hrs} \end{aligned}$$

Using Eq [14]:

$$\begin{aligned} dtd &= t_0 \times v_{def} \\ &= 0.3723 \text{ hrs} \times 30\text{kts} \\ &= 11.17\text{nm} \end{aligned}$$

Using Eq [20]:

$$\begin{aligned} \theta_{max} &= \sin^{-1} (dtd / cd) \\ &= \sin^{-1} (11.17\text{nm} / 33.83\text{nm}) \\ &= 19.28^\circ \end{aligned}$$

Hence, an increase in the radar detection range by 50nm (from 200nm to 250nm) yielded a 5.3° increase in θ_{max} from 13.98° (Exercise 1) to 19.28° . The resulting maximum threat coverage sector by a single defender which is $2 \times \theta_{max}$ is now 38.56° vice 20° (Exercise 1). Subsequently, the HVU now only requires 10 defenders for 360°

coverage without overlap. This is a reduction by 3 defenders from our original case with r_{det} equal to 200nm.

D. EFFECT OF LATERAL SEPARATION AMONG HVUS

EXAMPLE 5: To determine the effect of lateral separation of HVUs on threat sector and HVU placement

Given:

Attacker speed	500 kts
Defender speed	30 kts
Detection range	200 nm
Weapons range	30 nm
Threat Sector	$20^\circ (\theta = 10^\circ)$
Lateral Separation	10 nm
cd	33.83 nm

Since the first four parameters are the same as in Exercise 1, we will use some results from Exercise 1 to avoid unnecessary recalculations. The new parameter in this scenario is for the HVUs to maintain at least 10nm lateral separation from each other. We shall examine the effect of this new requirement on threat sector coverage capabilities of defenders, and on secondary HVU placement.

$$\begin{aligned} \text{Using Eq [22]: } \gamma &= \arcsin [(ls / 2) / r_{det}] \\ &= \arcsin [(10\text{nm} / 2) / 200\text{nm}] \\ &= 1.43^\circ \end{aligned}$$

$$\begin{aligned} \text{Using Eq [25]: } x &= \sin \gamma \times (r_{det} - cd) && *cd \text{ value from Exercise 1} \\ &= \sin (1.43^\circ) \times (200\text{nm} - 33.83\text{nm}) \\ &= 4.15\text{nm} \end{aligned}$$

$$\begin{aligned} \text{Using Eq [26]: } \tau &= \arcsin (x / cd) \\ &= \arcsin (4.15\text{nm} / 33.83\text{nm}) \end{aligned}$$

$$= 7.05^\circ$$

Using Eq [27]:

$$\begin{aligned}\Phi &= \theta + 2\tau \\ &= 10^\circ + 2(7.05^\circ) \\ &= 24.10^\circ\end{aligned}$$

Assuming symmetry on either side of the primary HVU, the defender must now be able to defend the *effective* threat sector which is 2Φ or 48.2° for at least 3 HVUs.

E. DETERMINING SALVO REQUIREMENTS

EXAMPLE 6: To determine the number of salvos required for a certain level of P_k against HDT

Given: HDT of 10 attackers

Defender has a three round salvo capability

$$p = 0.6$$

Our state transition matrix A is as shown below.

Salvo 1	10T	9T	8T	7T	6T	5T	4T	3T	2T	1T	0T
10T	0.064	0.288	0.432	0.216	0	0	0	0	0	0	0
9T	0	0.064	0.288	0.432	0.216	0	0	0	0	0	0
8T	0	0	0.064	0.288	0.432	0.216	0	0	0	0	0
7T	0	0	0	0.064	0.288	0.432	0.216	0	0	0	0
6T	0	0	0	0	0.064	0.288	0.432	0.216	0	0	0
5T	0	0	0	0	0	0.064	0.288	0.432	0.216	0	0
4T	0	0	0	0	0	0	0.064	0.288	0.432	0.216	0
3T	0	0	0	0	0	0	0	0.064	0.288	0.432	0.216
2T	0	0	0	0	0	0	0	0	0.16	0.48	0.36
1T	0	0	0	0	0	0	0	0	0	0.4	0.6
0T	0	0	0	0	0	0	0	0	0	0	1

The five-salvo transition matrix is A^5 :

Salvo 5	10T 0K	9T 1K	8T 2K	7T 3K	6T 4K	5T 5K	4T 6K	3T 7K	2T 8K	1T 9K	0T 10K
10T 0K	2.8E-10	1E-08	1.7E-07	1.9E-06	1.5E-05	9.1E-05	0.00043	0.00166	0.00921	0.09189	0.89669
9T 1K	0	2.8E-10	1E-08	1.7E-07	1.9E-06	1.5E-05	9.1E-05	0.00043	0.00347	0.0579	0.9381
8T 2K	0	0	2.8E-10	1E-08	1.7E-07	1.9E-06	1.5E-05	9.1E-05	0.00115	0.03482	0.96392
7T 3K	0	0	0	2.8E-10	1E-08	1.7E-07	1.9E-06	1.5E-05	0.00034	0.02033	0.97931
6T 4K	0	0	0	0	2.8E-10	1E-08	1.7E-07	1.9E-06	9.3E-05	0.01162	0.98828
5T 5K	0	0	0	0	0	2.8E-10	1E-08	1.7E-07	2.3E-05	0.00657	0.9934
4T 6K	0	0	0	0	0	0	2.8E-10	1E-08	5.8E-06	0.00378	0.99621
3T 7K	0	0	0	0	0	0	0	2.8E-10	1.3E-06	0.00196	0.99804
2T 8K	0	0	0	0	0	0	0	0	4.3E-07	0.00131	0.99869
1T 9K	0	0	0	0	0	0	0	0	0	0.00066	0.99934
0T 10K	0	0	0	0	0	0	0	0	0	0	1

Thus, in a scenario with a HDT of 10 attackers, a defender with a 3 round salvo capability, and a single shot probability of kill of 0.6, a minimum of 5 salvos are required for an approximately 90% probability of successfully killing all ten attackers. Similarly, if the HDT is comprised of only 8 attackers, the defender has a 96.39% chance of killing all 8 attackers. Further comparisons can easily be done by varying the defender's maximum shot capacity per salvo and the number of attackers in the HDT.

VII. MODEL LIMITATIONS AND AREAS FOR FURTHER ANALYSIS

A. DIRECT OVERFLIGHT OF ATTACKERS

Calculations were based on the assumption that the defender is able to position itself such that the attacker directly overflies the defender. Of course, with modern air defense ranges well beyond 50 nm or more, strict overhead flight is not a requirement. In such cases, a modification can be made to the engagement area, so that calculations for probability of kill take into consideration angled approaches across the engagement zone. *Naval Operational Analysis* discusses engagement of targets when not flying directly overhead.¹⁴

B. CALCULATIONS FOR MULTIPLE DEFENDERS

A defender may be able to engage attackers beyond the effective coverage sector, albeit at less than the maximum level. Since multiple defenders need not exclusively engage attackers, these degraded but overlapping coverage sectors by two or more defenders could also achieve an effective defense.

¹⁴ Daniel H. Wagner, *et. al.* ed., Naval Operations Analysis, Annapolis, Maryland: Naval Institute Press, 1999, ch. 13

Separately, the two defenders have a 40° *effective* coverage area.

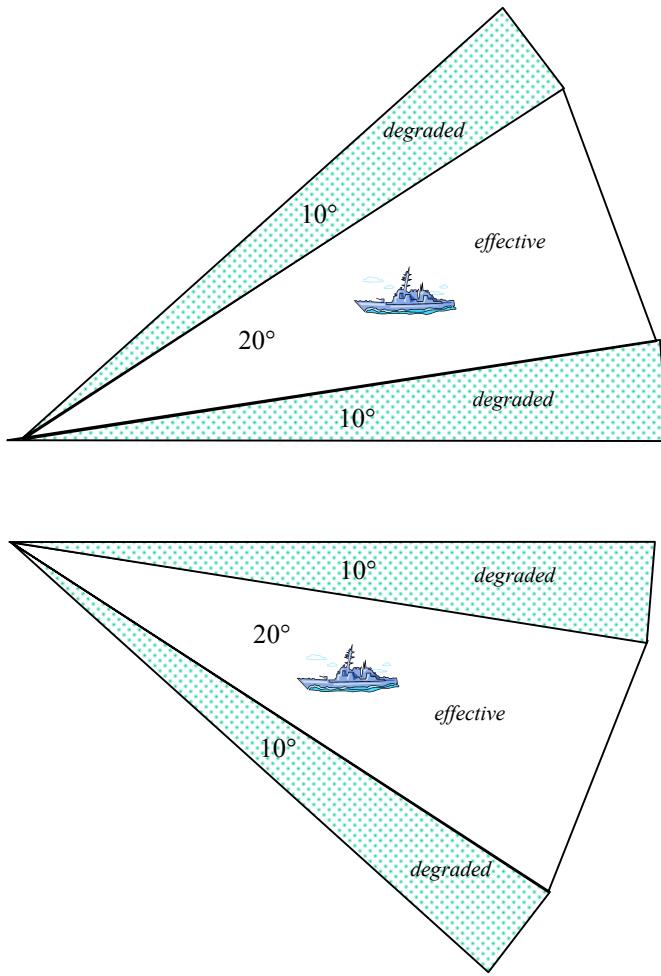


Figure 28 – Separate Coverage Zones

Combined, the two defenders have a 50° *effective* coverage area (Figure 29). Although this model does not currently take into account the effects of overlapping degrade sectors, further model development may examine this aspect.

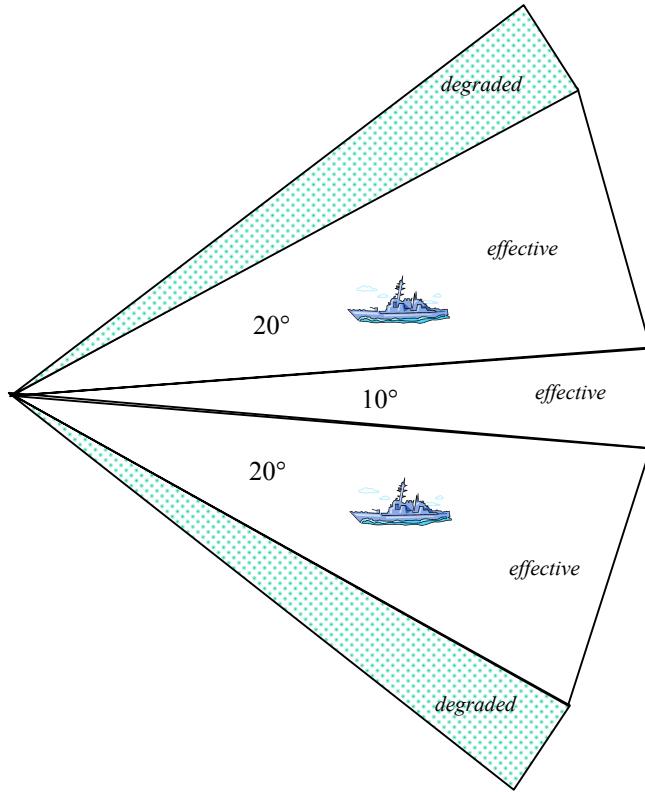


Figure 29 – Combined Coverage Zone

C. MOVING HVUS

Although there may be situations where the HVUs (i.e. EXWAR ship) must stay relatively stationary during sea based operations such as launching and recovering aircraft or watercraft, it is conceivable that the HVUs will have some freedom of mobility most of the time. This model could be modified to accommodate HVU movement by also calculating a HVU travel distance (htd) in a similar manner as the defender travel distance (dtd). If it is assumed that the HVU moves directly away from the direction of the threat's approach, this could possibly shorten the time/distance problem for the defender to reach the critical distance (cd) and allow greater flexibility in threat sector coverage.

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VIII. CONCLUSION

As expeditionary warfare takes on greater prominence in naval operations, the defense of the sea base from high density threats becomes vital. Current tactics and weapon systems are focused on relatively small raids. Thus, this paper examines the critical factors that contribute to the defense of the Sea Base with respect to multiple HVUs against HDTs. The paper focuses primarily on the following factors:

- (1) defender placement relative to threat and HVU positions,
- (2) defender weapons speed,
- (3) defender weapons inventory,
- (4) defender weapons range,
- (5) defender weapons single shot probability of kill,
- (6) HDT size,
- (7) HDT speed,
- (8) threat sector size,
- (9) number of HVUs, and
- (10) placement of HVUs relative to threat and defenders positions.

From a development standpoint, the calculations developed in chapters III-V can be used by Systems Engineers to perform trade off studies during ship or weapon system production. From an operational standpoint, the calculations can be used by a commander charged with the Sea Base defense to utilize his given assets in the most efficient manner. For example, the Wide Beam missile firing policy algorithm in Chapter V could be programmed into a defender's combat system to respond automatically with the appropriate salvo size or provide a recommendation to a commander for asset allocation.

The study is divided into three phases, each building on the previous phase. Phase I considers the scenario with a single HVU, a single defender, and a single attacker. This phase describes the fundamental relationships that comprise Sea Base defense and provides a methodology to determine the maximum sector a single defender can cover within the constraints of factors 1, 2, 4, 7, and 8. Phase II explores the effect of having multiple HVUs on defense of the Sea Base with particular emphasis on threat sector

coverage and HVU placement. Finally, Phase III examines the effect of having multiple attackers on Sea Base defense. In particular, the third phase looks at current capabilities and limitations of weapon systems and tactics against HDTs through a comparative analysis with a modified weapons system and tactic. The current system is a narrow beam FC radar. The modified system is a FC radar designed for a wide beam illumination of multiple targets. The Wide Beam algorithm in conjunction with this modified system allows a greater expected number of kills against the HDT. However, this gain is at a cost of potentially wasting missile due to possible redundant kills. Although the cost of a defender such as a cruiser or destroyer is in the billions of dollars, not to mention the invaluable human assets aboard, missiles are not inexpensive and are limited. Thus, the benefits of the Wide Beam model must be tempered against the risk of inefficient use of valuable assets.

XI. APPENDIX A – EXPECTED VALUE AND SIMULATION CODE

The following is written in Java.

```
/*
 * ExpectedValue.java
 *
 * Create December 4, 2003
 */

/**
 *
 * Henry S. Kim
 * LT USN
 * Comments: This class determines the expected number of killed attackers
 * and calculates the optimal firing policy given at each salvo:
 *
 * number of attackers, number of missiles in the defender's
 * inventory, number of salvos, and single-shot probability of kill.
 * In addition, this class calculates the optimal firing policy
 */

public class ExpectedValue {

    public static void main(String[] args) {
        int s; //variable for salvos left
        int w; //variable for weapons left
        int a; //variable for current number of attackers
        int salvos; //total potential salvos
        int weapinvent; //weapon's inventory
        int attackers; //initial number of attackers
        int y; //number of remaining attackers
        int x; //number of weapons assigned
        int i; //index for number of weapons on first attacker
        double sumofy;
        double TempEV;
        double roundEV;
        double [[[ EV;
        double [[[ pfunc;
        int [[[ StoreX;
        boolean seeoutput;
        boolean dofunc;
        boolean dowide;
        boolean dosim;
        boolean seeEVresults;
```

```

boolean seeprobdata;
boolean seesimoutput;
boolean seextable;

double p;
double q; // q = 1-p
double qi;
double ib;
double fac;
double pr;
double initterm;

salvos=5;
weapinvent=25;
attackers=10;
p= 0.75;

dofunc = true; // performs the expected value function
dowide = true; // if true perform wide beam, if false does narrow
seeoutput = false; // displays intermediate output for debugging
seeprobdata = false; // displays probability function results
seeEVresults = false; // displays f function results and associated firing
                     // recommendation
seextable = true; // displays firing recommendation in table format

dosim = false; // if true, will run simulation
seesimoutput = true; // displays intermediate simulation output for debugging

EV = new double[salvos+1][weapinvent+1][attackers+1];
pfunc = new double[attackers+1][weapinvent+1][attackers+1];
StoreX = new int[salvos+1][weapinvent+1][attackers+1];

//////// FILLING THE PROBABILITY TABLE //////

for (a=attackers;a>=0;a--) {
    for (x=weapinvent;x>=0;x--) {
        for (y=attackers;y>=0;y--) {
            if(a==y)
                pfunc[a][x][y] = 1;
            else
                pfunc[a][x][y] = 0;
        }
    }
}
if (p>0 && weapinvent>0){

```

```

q = 1.0 - p;
for (a=1;a<=attackers;a++){
    for (w=1;w<=weapinvent;w++){
        for (y=0;y<=a;y++){
            if(w+y>=a){
                if(a==1){
                    pfunc[1][w][0]=1-Math.pow(q,w);
                    pfunc[1][w][1]=Math.pow(q,w);
                }
                else{
                    fac=1.0/(a-1.0);
                    pr = Math.pow((1.0-(1.0/a)),w);
                    qi = 1.0;
                    ib=w;
                    if (y==0) {
                        sumofy=0;
                    }
                    else {
                        sumofy=pr*pfunc[a-1][w][y-1];
                    }
                    for (i = 1;i<=w;i++){
                        pr = pr * fac * ib / i;
                        qi = qi * q;
                        if(y>0) {
                            sumofy += pr*(qi*pfunc[a-1][w-i][y-1]+
                                (1.0-qi)*pfunc[a-1][w-i][y]);
                        }
                        else {
                            sumofy += pr*(1-qi)*pfunc[a-1][w-i][y];
                        }
                        ib=ib-1;
                    }
                    pfunc[a][w][y]=sumofy;
                }
            }
            if (seeprobdatal==true) {
                System.out.println("pfunc[a=" + a + "][w=" + w + "][y=" + y + "]:"
                    + pfunc[a][w][y]);
            }
        }
        if (seeprobdatal==true) {
            System.out.println();
            System.out.println();
        }
    }
}

```

```
}
```

```
////////// CALCULATION OF FUNCTION EV//////////
```

```
if (dofunc==true) {
```

```
////////// INITIALIZING for s = 0 //////////
```

```
s=0;
```

```
for (w=weapinvent;w>=1;w--) {
```

```
    for (a=attackers;a>=1;a--) {
```

```
        EV[s][w][a] = 0;
```

```
    }
```

```
}
```

```
//////////
```

```
for (s=1;s<=salvos;s++) {
```

```
    for (w=weapinvent;w>=1;w--) {
```

```
        for (a=attackers;a>=0;a--) {
```

```
            for (x=w;x>=0;x--) {
```

```
                sumofy = 0;
```

```
                for (y=a;y>=0;y--) {
```

```
                    if (dowide==true) {
```

```
                        sumofy += pfunc[a][x][y]*( (a-y) + EV[s-1][w-x][y]);
```

```
                    }
```

```
                    else {
```

```
                        if (y==a) {
```

```
                            sumofy += Math.pow((1-p),x)*( (a-y) + EV[s-1][w-x][y]);
```

```
                        }
```

```
                        if (y==(a-1)) {
```

```
                            sumofy += (1-(Math.pow((1-p),x)))*( (a-y) + EV[s-1][w-x][y]);
```

```
                        }
```

```
}
```

```
                    if (seeoutput == true) {
```

```
                        System.out.println("s: "+s+" w: "+w+" a: "+a+" x: "+x+" y: "+y);
```

```
                        System.out.println("sumofy: "+sumofy);
```

```
}
```

```
}
```

```
TempEV = sumofy;
```

```
if (seeoutput == true) {
```

```
    System.out.println("TempEV: " +TempEV);
```

```
}
```

```
if (EV[s][w][a] < TempEV) {
```

```
    if (seeoutput == true) {
```

```

        System.out.println("did EV[s][w][a]");
    }
    EV[s][w][a] = TempEV;
    StoreX[s][w][a] = x;
}
}
}
}

////////// PRINTING OUTPUT /////////////
if (seeEVresults==true) {
    for (s=1;s<=salvos;s++) {
        for (w=weapinvent;w>=0;w--) {
            for (a=attackers;a>=0;a--) {
                roundEV=Math.round(EV[s][w][a]*100.00)/100.00;
                System.out.println("EV[s="+s+"] [w="+w+"] [a="+a+"] : "+roundEV+
                    X[s+s+"] [w=w+] [a=a+] : "+StoreX[s][w][a]);
            }
            System.out.println();
        }
        System.out.println();
        System.out.println();
    }
}
if (seetable==true) {
    for (s=salvos;s>=1;s--) {
        System.out.println("RemainingSalvos:" + s);
        System.out.println();
        for (i=0;i<=attackers;i++) {
            if (i==attackers) {
                System.out.println(i);
            }
            else {
                System.out.print(i + " ");
            }
        }
        for (w=1;w<=weapinvent;w++) {
            System.out.print(w + " ");
            for (a=1;a<=attackers;a++) {
                if (a==attackers) {
                    System.out.println(StoreX[s][w][a]);
                } else {
                    System.out.print(StoreX[s][w][a] + " ");
                }
            }
        }
    }
}

```

```
        }  
        System.out.println();  
    }  
}
```

```
int t; // variable for trials
```

```

int stages; // variable for number of remaining engagement opportunities
int cankill; // possible killed attackers after each stage
int killed; // variable for number of attackers killed
int trials; // number of trials
int shoot; // recommended number of missiles to shoot
int totalkilled; // total killed at each stage
int cumulative; // cumulative killed after t trial runs
double avg; // average killed after t trials
double rn; // random number from 0 to 1
double [] limit; // limits for probability intervals

limit = new double [weapinvent+1];
trials = 2;

avg = 0.0;
if (dosim==true) {
    System.out.println("RUNNING SIMULATION WITH "+trials+" TRIALS");
    cumulative = 0;
    for (t=1; t<=trials;t++) {
        System.out.println(" ** TRIAL "+t+" **");
        totalkilled = 0;
        w=weapinvent;
        a=attackers;
        for (s=salvos; s>=1; s--) {
            shoot = StoreX[s][w][a];
            if (seesimoutput==true) {
                System.out.println(" --- Salvo "+(salvos+1-s)+" ---");
                System.out.println("attackers: "+a+" weapon inventory: "+w);
                System.out.println("recommend shooting: "+shoot);
            }
            cankill = Math.min(shoot, a);
            if (seesimoutput==true) {
                System.out.println("cankill: "+cankill);
            }
            for (killed=cankill;killed>=0;killed--) {
                if (killed==cankill) {

```

```

        if (seesimoutput==true) {
            System.out.println("potential decrease to: "+(a-killed));
        }
        limit[killed+1] = 1.0;
    }
    limit[killed] = limit[killed+1] - pfunc[a][shoot][a-killed];
}
rn = Math.random();
for (int interval=0; interval<=cankill; interval++) {
    if ( (rn >= limit[interval]) && (rn < limit[interval+1]) ) {
        a -= interval;
        if (seesimoutput==true) {
            System.out.println("we now fire "+shoot+" missiles");
            System.out.println(interval+" attackers were killed");
            System.out.println(" New number of attackers is: "+a);
        }
        totalkilled += interval;
    }
}
w -= shoot;
}
cumulative += totalkilled;
if (seesimoutput==true) {
    System.out.println();
    System.out.println("total killed on trial "+t+": "+totalkilled);
    System.out.println();
}
}
System.out.println();
System.out.println("cumulative: "+cumulative+" trials: "+trials);
avg = (double)cumulative/trials;
System.out.println("avg: "+avg);
System.out.println("EV[s='"+salvos+"'][w='"+weapinvent+"'][a='"+attackers+"']: "+EV[salvos][weapinvent][attackers]);
System.out.println();
}
}

```

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